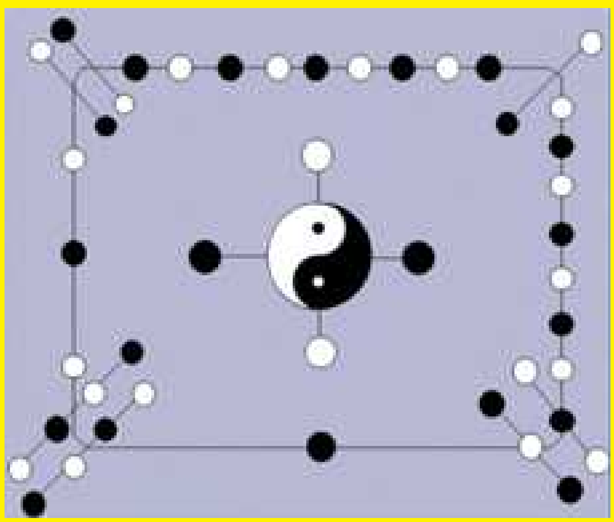
ISBN 978-1-59973-164-3

VOLUME 3, 2011

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES

September, 2011

Vol.3, 2011 ISBN 978-1-59973-164-3

Mathematical Combinatorics

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Even when the experts all agree, they may well be mistaken.

By Bertrand Russell, a Welsh philosopher, logician and mathematician.

Math.Combin.Book Ser. Vol.3 (2011), 01-10

Incidence Algebras and Labelings of Graph Structures

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Abstract: Ancykutty Joseph, *On Incidence Algebras and Directed Graphs*, IJMMS, 31:5(2002), 301-305, studied the incidence algebras of directed graphs. We have extended it to undirected graphs also in our earlier paper. We established a relation between incidence algebras and the labelings and index vectors introduced by R.H. Jeurissen in *Incidence Matrix and Labelings of a Graph*, Journal of Combinatorial Theory, Series B, Vol 30, Issue 3, June 1981, 290-301, in that paper. In this paper, we extend the concept to graph structures introduced by E. Sampathkumar in On Generalized Graph Structures, Bull. Kerala Math. Assoc., Vol 3, No.2, Dec 2006, 65-123.

Key Words: Graph structure, R_i-labeling, R_i-index vector, labelling matrix, index matrix, incidence algebra.

AMS(2010): 05C78, 05C50, 05C38, 06A11

§1. Introduction

Ancykutty Joseph introduced the concept of incidence algebras of directed graphs in [1]. She used the number of directed paths from one vertex to another for introducing the incidence algebras of directed graphs. Stefan Foldes and Gerasimos Meletiou [10] has discussed the incidence algebras of pre-orders also. This motivated us in our study on the incidence algebras of undirected graphs in [8]. We used the number of paths for introducing the concept of incidence algebras of undirected graphs. We also established a relation between incidence algebras and the labelings and index vectors of a graph as given by Jeurissen [12](based on the works of Brouwer [2], Doob [9] and Stewart [15]) in that paper.

E. Sampathkumar introduced the concept of a graph structure in [13] as a generalization of signed graphs. In this paper, we extend the results of our paper on graphs to graph structures and prove that the collection of all R_i-labelings for the collection of all admissible R_i- index vectors, the collection of all R_i-labelings for the index vector 0 and the collection of all R_i-labelings for the index vector $\lambda_i j_i$, ($\lambda_i \in F, F$, a commutative ring j_i an all 1-vector) of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$ are subalgebras of the incidence algebra $I(V, F)$. We also

[1]Received February 15, 2011. Accepted August 2, 2011.

prove that the set of labeling matrices for all admissible index matrices of a graph structure is a subalgebra of $I(V^k, F^k)$.

§2. Preliminaries

Throughout this paper, by a ring we mean an associative ring with identity. First We go through the definitions of commutative ring, partially ordered set, pre-ordered set etc. The following definitions are adapted from [16].

Definition 2.1 *A (left) A-module is an additive abelian group M with the operation of (left) multiplication by elements of the ring A that satisfies the following properties.*

(*i*) $a(x+y) = ax + ay$ *for any* $a \in A, x, y \in M$;

(*ii*) $(a+b)x = ax + bx$ *for any* $a, b \in A, x \in M$;

(*iii*) $(ab)x = a(bx)$ *for any* $a, b \in A, x \in M$;

(*iv*) $1x = x$ *for an* $x \in M$.

By an A-module, we mean a left A-module.

Definition 2.2 *A set* $\{x_1, x_2, ..., x_n\}$ *of elements of* M *is a basis for* M *if*

(*i*) $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$ *for* $a_i \in A$ *only if* $a_1 = a_2 = ... = a_n = 0$ *and*

(*ii*) M *is generated by* $\{x_1, x_2, ..., x_n\}$*, i.e.,* M *is the collection of all linear combinations of* $\{x_1, x_2, ..., x_n\}$ *with scalars from* A.

A finitely generated module that has a basis is called free.

Definition 2.3 *An algebra A is a set over a field K with operations of addition, multiplication and multiplication by elements of K that have the following properties.*

(*i*) *A is a vector space with respect to addition and multiplication by elements of the field.*

(*ii*) *A is a ring with respect to addition and multiplication.*

iii. $(\lambda a)b = a(\lambda b) = \lambda(ab)$ *for any* $\lambda \in K, a, b \in A$.

A subset S of an algebra A is called a subalgebra if it is simultaneously a subring and a subspace of A.

Definition 2.4([14]) *A set* X *with a binary relation* $\leq$ *is a pre-ordered set if* $\leq$ *is reflexive and transitive. If* $\leq$ *is reflexive, transitive and antisymmetric, then* X *is a partially ordered set (poset).*

E. Spiegel and C.J. O'Donnell [14] defined incidence algebra as follows.

Definition 2.5([14]) *The incidence algebra* $I(X, R)$ *of the locally finite partially ordered set* X *over the commutative ring* R *with identity is* $I(X, R) = \{f : X \times X \to R | f(x, y) =$

0 *if x is not less than or equal to y\}* *with operations given by*

$$\begin{aligned}(f+g)(x,y) &= f(x,y)+g(x,y)\\ (f.g)(x,y) &= \sum_{x\le z\le y} f(x,z).g(z,y)\\ (r.f)(x,y) &= r.f(x,y)\end{aligned}$$

for $f,g \in I(X,R)$ with $r \in R$ and $x,y,z \in X$.

Ancykutty Joseph [1] established a relation between incidence algebras and directed graphs. The incidence algebra $I(G,Z)$ for digraph without cycles and multiple edges $(G,\le)$ representing the finite poset $(V,\le)$ is defined in [1] as follows.

Definition 2.6([1]) *For $u,v \in V$, let $p_k(u,v)$ denote the number of directed paths of length k from u to v and $p_k(v,u) = -p_k(u,v)$. For $i = 0,1,\cdots,n-1$, define $f_i, f_i^* : V\times V \to Z$ by $f_i(u,v) = p_i(u,v), f_i^*(u,v) = -p_i(u,v)$. The incidence algebra $I(G,Z)$ of $(G,\le)$ over the commutative ring Z with identity is defined by $I(G,Z) = \{f_i, f_i^* : V\times V \to Z, i = 0,1,...,n-1\}$ with operations defined as*

(i) *For $f \neq g$,$(f+g)(u,v) = f(u,v)+g(u,v)$;*
(ii) $(f.g)(u,v) = \sum_w f(u,w)g(w,v)$;
(iii) $(zf)(u,v) = z.f(u,v)\forall z \in Z; f,g \in I(G,Z)$.

In [10], Stefan Foldes and Gerasimos Meletiou says about incidence algebra of pre-order as follows.

Definition 2.7([10]) *Given a field F, the incidence algebra $A(\rho)$, of a pre-ordered set $(S,\rho), S = \{1,2,...,n\}$ over F is the set of maps $\alpha : S^2 \to F$ such that $\alpha(x,y) = 0$ unless $x\rho y$. The addition and multiplication in $A(\rho)$ are defined as matrix sum and product.*

Replacing field F by a commutative ring R with identity and following the definition of Foldes and Meletiou[10], we obtained in graphs [8] an analogue of the incidence algebra of a directed graph given by Ancykutty Joseph[1].

Theorem 2.1([8]) *Let $G = (V,E)$ be a graph without cycles and multiple edges with V and E finite. For $u,v \in V$, let $f_i(u,v)$ be the number of paths of length i between u and v. Then $\{f_i\}$ is an incidence algebra of (G,ρ) denoted by $I(G,Z)$ over the commutative ring Z with identity.*

§3. Graph Structure and Incidence Algebra

We recall some basic definitions on graph structure given by E. Sampathkumar[13].

Definition 3.1([13]) *$G = (V,R_1,R_2,\cdots,R_k)$ is a graph structure if V is a non empty set and $R_1,R_2,\cdots,R_k$ are relations on V which are mutually disjoint such that each $R_i, i = 1,2,\cdots,k$, is symmetric and irreflexive.*

If $(u,v) \in R_i$ for some $i, 1 \le i \le k$, (u,v) is an R_i-edge. R_i-path between two vertices u and v consists only of R_i-edges. G is $R_1R_2\cdots R_k$ connected if G is R_i-connected for each i.

We define $R_{i_1i_2\cdots i_r}$-path, $1 \le r \le k$, in a similar way as follows.

Definition 3.2 *A sequence of vertices $x_0.x_1, \cdots, x_n$ of V of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$ is an $R_{i_1i_2\cdots i_r}$-path, $1 \le r \le k$, if $R_{i_1}, R_{i_2}, \cdots, R_{i_r}$ are some among $R_1, R_2, \cdots, R_k$ which are represented in it.*

Note that the above definition matches with the concepts introduced in [4] by the authors.

Theorem 3.1 *Let $f_i^j(u,v)$ be the number of R_i-paths of length j between u and $f_i^{j*}(u,v) = -f_i^j(u,v)$. $I_{R_i}(G,Z) = \{f_i^j, f_i^{j*} : V \times V \to Z, i = 0, 1, ..., n-1\}$ is an incidence algebra over Z.*

Proof Let f_i^r and f_i^s be R_i-paths of length r and s respectively. For $f_i^r \ne f_i^s \in I_{R_i}(G,Z)$, define $((f_i^r + f_i^s)(u,v))$ = number of R_i-paths of length either r or s between u and v= $f_i^r(u,v) + f_i^s(u,v)$. Then

$$\begin{aligned}(f_i^r.f_i^s)(u,v) &= \text{number of } R_i\text{-paths of length } r+s \text{ between } u \text{ and } v\\ &= \sum_{w:(u,w)\in R_i,(w,v)\in R_i} f_i^r(u,w) f_i^s(w,v).\end{aligned}$$

$(zf_i^r)(u,v) = z.f_i^r(u,v) \forall z \in Z; f_i^r, f_i^s \in I_{R_i}(G,Z)$ (The operations are extended in the usual way if either or both are elements of the form f_i^{r*}).

So $I_{R_i}(G,Z)$ is an incidence algebra over Z. $\square$

Note 1. We may also consider another type of incidence algebras. Let $f^l_{i_1i_2\cdots i_r}(u,v)$ be the number of $R_{i_1i_2\cdots i_r}$ paths of length l between u and v and $f^{l*}_{i_1i_2\cdots i_r}(u,v) = -f^l_{i_1i_2\cdots i_r}(u,v)$. Then $I_{i_1i_2\cdots i_r}(V,Z) = \{f^l_{i_1i_2\cdots i_r}, f^{l*}_{i_1i_2\cdots i_r} : V \times V \to Z, i = 0, 1, \cdots, n-1\}$ with operations defined as follows is another subalgebra over Z.

(i) $(f^l_{i_1i_2\cdots i_r} + f^m_{i_1i_2\cdots i_r})(u,v) = f^l_{i_1i_2\cdots i_r}(u,v) + f^m_{i_1i_2\cdots i_r}(u,v)$.

(ii) $$(f^l_{i_1i_2\cdots i_r}.f^m_{i_1i_2\cdots i_r})(u,v) = \sum_{w:(u,w),(w,v)\in \bigcup_{i=i_1}^{i_r} R_i} f^l_{i_1i_2\cdots i_r}(u,w) f^m_{i_1i_2\cdots i_r}(w,v).$$

(iii) $(zf^l_{i_1i_2\cdots i_r})(u,v) = z.f^l_{i_1i_2\cdots i_r}(u,v) \forall z \in Z; f^l_{i_1i_2\cdots i_r}, f^m_{i_1i_2\cdots i_r} \in I_{i_1i_2\cdots i_r}(G,Z)$. (The operations are extended in the usual way if either or both are elements of the form f_i^{r*}).

Thus $I_{i_1i_2\ldots i_r}(V,Z)$ is an incidence algebra over Z.

Note 2. Another possibility is to consider a subalgebra consisting of various paths of the type $R_{i_1i_2\cdots i_r}$ with all of $i_1i_2\cdots i_r$ being different from $j_1j_2\cdots j_s$ for any two $u-v$ paths $f_{i_1i_2\cdots i_r}$ and $f_{j_1j_2\cdots j_s}$. Let $f^l_{l_1l_2\cdots l_r}, f^m_{m_1m_2\cdots m_s}$ be $R_{i_1i_2\cdots i_r}$ and $R_{j_1j_2\cdots,j_s}$-paths of length l and m respectively. Define

$$(f^l_{i_1i_2\cdots i_r} + f^m_{j_1j_2\cdots j_s})(u,v) = f^l_{i_1i_2\cdots i_r j_1j_2\cdots j_s}(u,v) + f^m_{i_1i_2\cdots i_r j_1j_2\cdots j_s}(u,v),$$

$$(f^j_{i_1 i_2 \cdots i_r}.f^j_{j_1 j_2 \cdots j_s})(u,v) = \sum_{w:(u,w),(w,v)\in \bigcup_{i=i_1}^{i_r} R_i} f^l_{i_1 i_2 \ldots i_r}(u,w) f^m_{j_1 j_2 \ldots j_s}(w,v),$$

$$(zf^l_{l_1 l_2 \cdots l_r})(u,v) = z.f^l_{l_1 l_2 \cdots l_r}(u,v),$$

$$I_{path}(V,Z) = \{f, f^* : V\times V \to Z\},$$

where f is an $R_{i_1 i_2 \cdots i_r}$-path, $i_1, i_2, \cdots, i_r \in \{1,2,\cdots,k\}, 1 \le r \le k$ and $f^* = -f$. (The operations are extended in the usual way if either or both are elements of the form f^*).

Thus $I_{path}(V,Z)$ is an incidence algebra over Z.

§4. R_i-labelings and Incidence Algebra

Now consider R_i-labelings and R_i-index vectors of G. We recall the concepts of R_i-labelings and R_i-index vectors introduced in [5].

Definition 4.1([5]) *Let F be an abelian group or a ring and $G = (V, R_1, R_2, \cdots, R_k)$ be a graph structure with vertices $v_0, v_1, \cdots, v_{p-1}$ and q_i number of R_i-edges. A mapping $r_i : V \to F$ is an R_i-index vector with components $r_i(v_0), r_i(v_1), \cdots, r_i(v_{p-1}), i = 1,2,\cdots,k$ and a mapping $x_i : R_i \to F$ is an R_i-labeling with components $x_i(e_i^1), x_i(e_i^2), \cdots, x_i(e_i^{q_i}), i = 1,2,\cdots,k$.*

An R_i-labeling x_i is an R_i-labeling for the R_i-index vector r_i iff $r_i(v_j) = \sum_{e_r\in E_i^j} x_i(e_r)$, where E_i^j is the set of all R_i-edges incident with v_j. R_i-index vectors for which an R_i-labeling exists are called admissible R_i-index vectors.

Now we prove some results on R_i-labellings and incidence algebras. For that, first we recall the operations of addition and scalar multiplication mentioned in [5].

$$(r_i^1 + r_i^2)(v_j) = r_i^1(v_j) + r_i^2(v_j),$$
$$(fr_i^1)(v_j) = fr_i^1(v_j),$$
$$(x_i^1 + x_i^2)(e_j) = x_i^1(e_j) + x_i^2(e_j),$$
$$(fx_i^1)(e_j) = fx_i^1(e_j).$$

Now we define multiplication as follows.

Definition 4.2 *Let r_i^1, r_i^2 be R_i-index vectors and x_i^1, x_i^2 be R_i-labelings of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$.*

$$(r_i^1.r_i^2)(v_l) = \sum_{s:(v_l,v_s)\in R_i} r_i^1(v_l) r_i^2(v_s)$$

$$(x_i^1.x_i^2)(v_l,v_m) = 2.\sum_{s:(v_l,v_s)\in R_i,(v_s,v_m)\in R_i} x_i^1(v_l,v_s)x_i^2(v_s,v_m)$$ *(Multiplication by 2 is to adjust the duplication due to symmetric property of R_i-edges).*

Now we prove that with respect to these operations, the set of all R_i-labelings for all admissible R_i-index vectors is a subalgebra of the incidence algebra $I(V,F)$.

Theorem 4.1 *The set of R_i-labelings for all admissible R_i-index vectors of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$ is a subalgebra of $I_{L(A_i)}(V, F)$ where A_i is the collection of all admissible R_i-index vectors.*

Proof Let $I_{L(A_i)}(V, F)$ be the collection of R_i-labelings for elements of A_i. Let $x_i^1, x_i^2 \in I_{L(A_i)}(V, F)$. Then there exist $r_i^1, r_i^2 \in F$ such that

$$r_i^1(v_j) = \sum_{p:(v_j,v_p)\in R_i} x_i^1(v_j, v_p) \text{ and } r_i^2(v_j) = \sum_{p:(v_j,v_p)\in R_i} x_i^2(v_j, v_p).$$

$$\begin{aligned}(r_i^1 + r_i^2)(v_j) &= r_i^1(v_j) + r_i^2)(v_j) = \sum_{p:(v_j,v_p)\in R_i} x_i^1(v_j, v_p) + \sum_{p:(v_j,v_p)\in R_i} x_i^2(v_j, v_p) \\ &= \sum_{p:(v_j,v_p)\in R_i} (x_i^1 + x_i^2)(v_j, v_p).\end{aligned}$$

Therefore $x_i^1 + x_i^2$ is an R_i-labeling for $(r_i^1 + r_i^2$, i.e., $x_i^1 + x_i^2 \in I_{L(A_i)}(V, F)$.

$$\begin{aligned}(ri^1.r_i^2)(v_j) &= \sum_{s:(v_jv_s)\in R_i} r_i^1(v_j) r_i^2(v_s) \\ &= \sum_{s:(v_jv_s)\in R_i} [\sum_{l:(v_jv_l)\in R_i} x_i^1(v_j, v_l) \sum_{m:(v_sv_m)\in R_i} x_i^2(v_s, v_m)] \\ &= 2. \sum_{s:(v_jv_s)\in R_i} \sum_{m:(v_sv_m)\in R_i} x_i^1(v_j, v_s) x_i^2(v_s, v_m) \\ &= \sum_{n:(v_jv_n)\in R_i} (x_i^1 x_i^2)(v_j, v_n)\end{aligned}$$

Therefore $x_i^1.x_i^2$ is an R_i-labeling for $r_i^1.r_i^2$. i.e., $x_i^1.x_i^2 \in I_{L(A_i)}(V, F)$.

$$\begin{aligned}(fr_i^1)(v_j) &= f.r_i^1(v_j) \\ &= f. \sum_{n:(v_jv_n)\in R_i} x_i^1(v_j, v_n) \\ &= \sum_{n:(v_jv_n)\in R_i} fx_i^1(v_j, v_n) \\ &= \sum_{n:(v_jv_n)\in R_i} (fx_i^1)(v_j, v_n)\end{aligned}$$

i.e., $fx_i^1 \in I_{L(A_i)}(V, F)$. Hence $I_{L(A_i)}(V, F)$ is a subalgebra of $I(V, F)$. □

For the next few results, we require results from our previous papers [5] and [7].

Theorem 4.2([5]) *If F is an integral domain, the R_i-labelling of G for the R_i-index vector 0 form a free F-module.*

Theorem 4.3([7]) *Let F be an integral domain. Then $S_i(G)$, the collection of R_i-labelings for $\lambda_i j_i, \lambda_i \in F, j_i$ an all 1-vector, is a free F-module.*

Theorem 4.4 *The set of R_i-labellings for $\lambda_i j_i, \lambda_i \in F, j_i$ an all 1 vector of a graph structure $G = (V, R_1, R_2, \cdots, R_k)$ forms a subalgebra of the incidence algebra $I(V, F)$.*

Proof Let $I_{L(\lambda_i)}(V,F)$ be the collection of R_i-labelings for $\lambda_i j_i$. Let $x_i^1, x_i^2 \in I_{L(\lambda_i)}(V,F)$. Then there exist $\lambda_i^1, \lambda_i^2 \in F$ such that

$$\lambda_i^1(v_j) = \sum_{p:(v_j v_p)\in R_i} x_i^1(v_j, v_p) \text{ and } \lambda_i^2(v_j) = \sum_{p:(v_j v_p)\in R_i} x_i^2(v_j, v_p).$$

By Theorem 4.3, $\lambda_i j_i$ is an F-module. Hence it is enough if we prove that $x_i^1.x_i^2$ is an R_i-labeling for $(\lambda_i^1.\lambda_i^2)j$

$$\begin{aligned}
(\lambda_i^1.\lambda_i^2)(v_j) &= \sum_{s:(v_j v_s)\in R_i} \lambda_i^1(v_j)\lambda_i^2(v_s) \\
&= \sum_{s:(v_j v_s)\in R_i} [\sum_{l:(v_j v_l)\in R_i} x_i^1(v_j, v_l) \sum_{m:(v_s v_m)\in R_i} x_i^2(v_s, v_m)] \\
&= 2. \sum_{s:(v_j v_s)\in R_i,(v_s v_n)\in R_i} x_i^1(v_j, v_s)x_i^2(v_s, v_n) \\
&= \sum_{n:(v_j v_n)\in R_i} (x_i^1 x_i^2)(v_j, v_n)
\end{aligned}$$

Therefore $x_i^1.x_i^2$ is an R_i-labeling for $\lambda_i^1.\lambda_i^2 = \lambda_i^3$. i.e., $x_i^1.x_i^2 \in I_{L(\lambda_i)}(V,F)$. Hence $I_{L(\lambda_i)}(V,F)$ is a subalgebra of $I(V,F)$. □

Theorem 4.5 *The set of R_i-labelings for 0 of a graph structure $G = (V, R_1, R_2, ..., R_k)$ forms a subalgebra of the incidence algebra $I(V,F)$.*

Let $I_{L(0_i)}$ be the collection of all R_i-labelings for 0. By Theorem 4.2, the collection is an F-module. So it is enough if we prove that $x_i^1.x_i^2 \in I_{L(0_i)}(V,F)\forall x_i^1, x_i^2 \in I_{L(0_i)}(V,F)$.

$$\begin{aligned}
\sum_{n:(v_j v_n)\in R_i} (x_i^1.x_i^2)(v_j, v_n) &= 2. \sum_{n:(v_j v_n)\in R_i} [\sum_{s:(v_j v_s)\in R_i,(v_s v_n)\in R_i} x_i^1(v_j, v_s)x_i^2(v_s, v_n)] \\
&= \sum_{s:(v_j v_s)\in R_i} x_i^1(v_j, v_s)[\sum_{n:(v_s v_n)\in R_i} x_i^2(v_s, v_n)] \\
&= \sum_{s:(v_j v_s)\in R_i} x_i^1(v_j, v_s).0(v_s) \\
&= 0
\end{aligned}$$

Therefore $x_i^1.x_i^2$ is an R_i-labeling for 0. ie., $x_i^1.x_i^2 \in I_{L(0_i)}(V,F)$. So $I_{L(0_i)}(V,F)$ is a subalgebra of $I(V,F)$. □

§5. Labeling Matrices and Incidence Algebras

We now establish the relation between labeling matrices and incidence algebras. For that first we recall the concepts of labeling matrices and index matrices of a graph structure introduced by the authors in [6].

Definition 5.1([6]) *Let F be an abelian group or a ring. Let R_i be an R_i-index vector and x_i be an R_i-labeling for $i = 1, 2, \cdots, k$. Then*

$$x = \begin{bmatrix} x_1 & 0 & . & . & . & 0 \\ 0 & x_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & . & x_k \end{bmatrix}$$

is a labeling matrix and

$$r = \begin{bmatrix} r_1 & 0 & . & . & . & 0 \\ 0 & r_2 & 0 & . & . & 0 \\ . & 0 & . & & & . \\ . & . & & . & & . \\ . & . & & & . & 0 \\ 0 & 0 & . & . & . & r_k \end{bmatrix}$$

is an index matrix for the graph structure $G = (V, R_1, R_2, \cdots, R_k)$.

$$x : \begin{bmatrix} R_1 \\ R_2 \\ . \\ . \\ . \\ R_k \end{bmatrix} \to F^k$$

is a labeling for $r : V^k \to F^k$ *if* $\sum_{m \in E_s} x_i(m) = r_i(x_s)$ *for* $s = 0, 1, \cdots, p-1; i = 1, 2, \cdots, k$. *If* r_i *is an admissible* R_i*-index vector* $i = 1, 2, \cdots, k$, *then* r *is called an admissible index matrix for* G.

Now we establish some relations between these and incidence algebras.

Theorem 5.1 *The set of labeling matrices for all admissible index matrices of a graph structure* $G = (V, R_1, R_2, \cdots, R_k)$ *is a subalgebra of* $I(V^k, F^k)$.

Proof Let $I_{L(A)}(V^k, F^k)$ be the set of all labeling matrices for the elements of A, the set of all admissible index matrices. Let $x_1, x_2 \in I_{L(A)}(V^k, F^k)$. Then $x_i^1, x_i^2 \in I_{L(A_i)}(V, F)$, the set of all R_i-labelings for the elements of the set A_i of all admissible R_i-index vectors for $i = 1, 2, \cdots, k$. Then as proved in Theorem 4.1, $x_i^1 + x_i^2, x_i^1.x_i^2, fx_i^1 \in I_{L(A_i)}(V, F)$ where $f \in F$. Hence $x^1 + x^2, x^1.x^2, fx^1$ are labelings for some $r^1 + r^2, r^1.r^2, fr^1$ respectively. i.e., $x^1 + x^2, x^1.x^2, fx^1 \in I_{L(A)}(V^k, F^k)$. So $I_{L(A)}(V^k, F^k)$ is a subalgebra of $I(V^k, F^k)$. □

Theorem 5.2 *The set of labeling matrices for* ΛJ *with*

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & . & . & . & 0 \\ 0 & \lambda_2 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ 0 & 0 & . & . & 0 & \lambda_k \end{bmatrix}, \quad J = \begin{bmatrix} j_1 & 0 & . & . & . & 0 \\ 0 & j_2 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ 0 & 0 & . & . & 0 & j_k \end{bmatrix},$$

j_i*, an all 1-vector for* $i = 1, 2, ..., k$ *of a graph structure* $G = (V, R_1, R_2, \cdots, R_k)$ *is a subalgebra of* $I(V^k, F^k)$*.*

Proof Let $I_{L(\Lambda)}(V^k, F^k)$ be the set of all labeling matrices for the index matrix Λ. Let $x_1, x_2 \in I_{L(\Lambda)}(V^k, F^k)$. Then $x_i^1, x_i^2 \in I_{L(\lambda_i)}(V, F)$, the set of all R_i-labellings for λ_i for $i = 1, 2, ..., k$. Then as proved in Theorem 4.4, $x_i^1 + x_i^2, x_i^1.x_i^2, fx_i^1 \in I_{L(\lambda_i)}(V, F)$ where $f \in F$. Hence $x^1 + x^2, x^1.x^2, fx^1$ are labelings for $\Lambda^1 + \Lambda^2, \Lambda^1.\Lambda^2, f\Lambda^1$ respectively, i.e.,$x^1 + x^2, x^1.x^2, fx^1 \in I_{L(\Lambda)}(V^k, F^k)$. So $I_{L(\Lambda)}(V^k, F^k)$ is a subalgebra of $I(V^k, F^k)$. □

Theorem 5.3 *The set of labeling matrices for* 0 *of a graph structure* $G = (V, R_1, R_2, ..., R_k)$ *is a subalgebra of* $I(V^k, F^k)$*.*

Proof Let $I_{L(0)}(V^k, F^k)$ be the set of all labeling matrices for the index matrix 0. Let $x_1, x_2 \in I_{L(0)}(V^k, F^k)$. Then $x_i^1, x_i^2 \in I_{L(0_i)}(V, F)$, the set of all R_i-labelings for 0 for $i = 1, 2, ..., k$. Then as proved in Theorem 4.5, $x_i^1 + x_i^2, x_i^1.x_i^2, fx_i^1 \in I_{L(0_i)}(V, F)$ where $f \in F$. Hence $x^1 + x^2, x^1.x^2, fx^1$ are labelings for $0 + 0 = 0, 0.0 = 0, f0 = 0$ respectively, i.e., $x^1 + x^2, x^1.x^2, fx^1 \in I_{L(0}(V^k, F^k)$. So $I_{L(0)}(V^k, F^k)$ is a subalgebra of $I(V^k, F^k)$. □

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Math.Combin.Book Ser. Vol.3 (2011), 11-16

Ideal Graph of a Graph

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Abstract: In this paper, we introduce ideal graph of a graph and study some of its properties. We characterize connectedness, isomorphism of graphs and coloring property of a graph using ideal graph. Also, we give an upper bound for chromatic number of a graph.

Key Words: Graph, Smarandachely ideal graph, ideal graph, isomorphism.

AMS(2010): 05C38, 05C60

§1. Introduction

Graphs considered here are finite, simple and undirected. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. Terms not defined here are used in the sense of Harary [2] and Gary Chartrand [1]. Two Graphs G_1 and G_2 are isomorphic if there exists a one-to-one correspondence f from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. By a coloring of a graph, we mean an assignment of colors to the vertices of G such that adjacent vertices are colored differently. The smallest number of colors in any coloring of a graph G is called the chromatic number of G and is denoted by $\chi(G)$. If it is possible to color G from a set of k colors, then G is said to be k-colorable. A coloring that uses k-colors is called a k-coloring.

§2. Ideal Graph of a Graph

In this section, we introduce ideal graph of a graph. We can analyze the properties of graphs by using ideal graph of a graph, which may be of smaller size than the original graph.

Definition 2.1 *For a graph G with sets $\mathscr{C}$ of cycles, $\mathscr{L}$ of longest paths with all the internal vertices of degree* 2*, and $U \subset \mathscr{C}$, $V \subset \mathscr{L}$, its Smarandachely ideal graph $I_d^{U,V}(G)$ of the graph*

[1]Received February 23, 2011. Accepted August 10, 2011.

G is formed as follows:

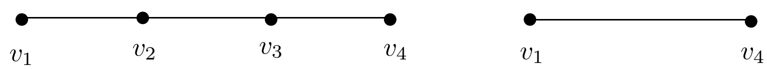
(i) *These cycles and the edges lying on a cycle in U or $\mathscr{C} \setminus U$ will remain or not same in Smarandachely ideal graph $I_d^{U,V}(G)$ of G.*

(ii) *Every longest u-v path in V or $\mathscr{L} \setminus V$ is considered as an edge uv or not in Smarandachely ideal graph $I_d^{U,V}(G)$ of G.*

Particularly, if $U = \mathscr{C}$ and $V = \mathscr{L}$, i.e., a Smarandachely $I_d^{\mathscr{C},\mathscr{L}}(G)$ of G is called the ideal graph of G, denoted by $I_d(G)$.

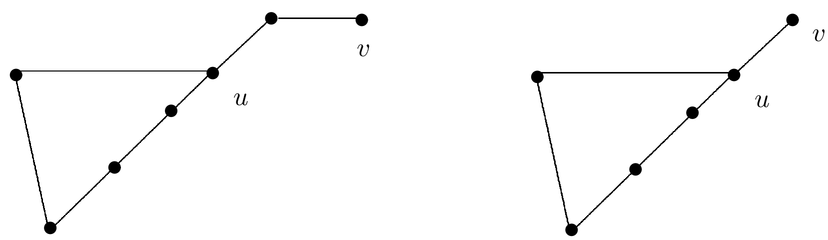
Example 2.2 Some ideal graphs of graphs are shown following.

1.



G $I_d(G)$

2.



G $I_d(G)$

Definition 2.3 *The vertices of the ideal graph $I_d(G)$ are called strong vertices of the graph G and the vertices, which are not in the ideal graph $I_d(G)$ are called weak vertices of the graph G.*

Definition 2.4 *The vanishing number of an edge uv of the ideal graph of a graph G is defined as the number of internal vertices of the u-v path in the graph G.*

We denote the vanishing number of an edge e of an ideal graph by $v_0(e)$.

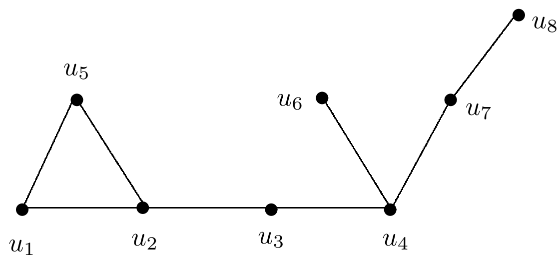
Remark 2.5 It is possible to get the original graph G from its ideal graph $I_d(G)$ if we know the vanishing numbers of all the edges of $I_d(G)$.

Definition 2.6 *The vanishing number of the ideal graph of a graph G is denoted by v_{id} and is defined as the sum of all vanishing numbers of the edges of $I_d(G)$ or the number of weak vertices of the graph G.*

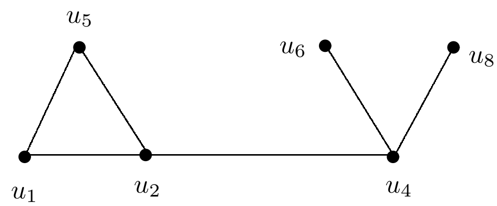
Definition 2.7 *The ideal number of a graph G is defined as the number of vertices in the ideal graph of the graph G or the number of strong vertices of the graph. It is denoted by p_{id}.*

Example 2.8 A graph with its ideal graph is shown in the following. In this graph, the ideal number of the graph G is 6. (i.e. $p_{id} = 6$). Also, in the ideal graph, the vanishing number of

the edges are $v_0(u_1u_2) = v_0(u_2u_5) = v_0(u_1u_5) = v_0(u_4u_6) = 0$ and $v_0(u_2u_4) = v_0(u_4u_8) = 1$. The vanishing number$(v_{id})$ of the ideal graph $I_d(G)$ is 2.



G



$I_d(G)$

The following proposition is obvious from the above definitions.

Proposition 2.9 *Let G be a graph and $p = |V(G)|$. The following properties are true.*

(i) $p = p_{id} + v_{id}$.

(ii) $p \geq p_{id}$.

(iii) $p = p_{id}$ *if and only if* $G = I_d(G)$.

Proof Proof follows from the Definitions 2.1, 2.6 and 2.7. □

Proposition 2.10 *There are ideal graphs following.*

(i) $I_d(P_n) = P_2$ for every $n \geq 2$.

(ii) $I_d(C_n) = C_n$, $I_d(W_n) = W_n$ and $I_d(K_n) = K_n$ for all n.

(iii) $I_d(K_{1,2}) = P_2$.

(iv) $I_d(K_{m,n}) = K_{m,n}$ except for $K_{1,2}$.

(v) $I_d(G) = G$ if $\delta \geq 3$.

(vi) $I_d(G) = G$ if G is Eulerian.

(vii) $I_d(I_d(G)) = I_d(G)$ for any graph G.

Proof Proof follows from the definition of $I_d(G)$. □

Proposition 2.11 *A vertex v of a graph G is a strong vertex if and only if $deg(v) \leq 1$ or $deg(v) \geq 3$ or the vertex v lies in a cycle.*

Proof Proof follows from the definition of $I_d(G)$. □

Proposition 2.12 *If a vertex v of a graph G is a weak vertex, then $deg(v) = 2$.*

proof Proof follows from the definition of $I_d(G)$. □

Remark 2.13 Converse of the above proposition is not true. For, consider $G = C_3$. Then all the vertices of G are of degree 2 but they are not weak vertices.

§3. Characterization of Connectedness

In this section, we characterize connected graphs using ideal graph.

Theorem 3.1 *A graph G is connected if and only if $I_d(G)$ is connected.*

Proof It is obvious from the definition of $I_d(G)$ that if G is connected, then $I_d(G)$ is connected. Assume that $I_d(G)$ is connected. Let u and v be two vertices of G.

Case i. u and v are strong vertices of G.

Since $I_d(G)$ is connected, there exists an u-v path in $I_d(G)$ that gives an u-v path in G.

Case ii. u is a strong vertex and v is a weak vertex of G.

Then v is an internal vertex of an u_1-v_1 path of G where u_1v_1 is an edge of $I_d(G)$. By assumption there exists an u-u_1 path in $I_d(G)$. Then the paths u-u_1 and u_1-v jointly gives the path in G between u and v.

Case iii. Both u and v are weak vertices of G.

Then u and v are internal vertices of some u_1-w_1 path and u_2-w_2 path in G respectively such that u_1w_1 and u_2w_2 are edges of $I_d(G)$. Then there exists an w_1-u_2 path in $I_d(G)$. Then the paths uw_1u_2v is the required u-v path in G. □

Theorem 3.2 *A graph G and $I_d(G)$ have same number of connected components.*

Proof Proof is obvious from the definition of $I_d(G)$ and Theorem 3.1.

§4. Characterization of Isomorphism

In this section, we characterize isomorphism of two graphs via ideal graphs. Since trees are connected graphs with no cycles, this characterization maybe more useful to analyze the isomorphism of trees.

Lemma 4.1([1]) *If a graph G is isomorphic to a graph G' under a function f, then*

(i) G and G' have same degree sequence

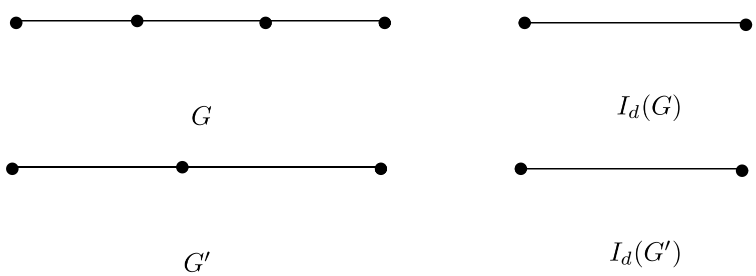
(ii) if G contains a k-cycle for some integer $k \geq 3$, so does G' and

(iii) if G contains a u-v path of length k, then G' contains a $f(u) - f(v)$ path of length k.

Theorem 4.2 *If a graph G is isomorphic to a graph G', then $I_d(G)$ is isomorphic to $I_d(G')$.*

Proof Proof follows from Lemma 4.1. □

Remark 4.3 The following example shows that the converse of the above theorem is not true.



Here, $I_d(G)$ and $I_d(G')$ are isomorphic. But G and G' are not isomorphic.

The following theorem gives the necessary and sufficient condition for two graphs to be isomorphic.

Theorem 4.4 *A graph G is isomorphic to the graph G' if and only if $I_d(G)$ is isomorphic to $I_d(G')$ and the isomorphic edges have same vanishing number.*

Proof Assume the graph G is isomorphic to the graph G'. By Theorem 4.2 and Lemma 4.1, $I_d(G)$ is isomorphic to $I_d(G')$ and the isomorphic edges have same vanishing number. Conversely, assume Id(G) is isomorphic to Id(G') and the isomorphic edges have same vanishing number. If uv and $u'v'$ are isomorphic edges of $I_d(G)$ and $I_d(G')$ respectively with same vanishing number, then the edges uv and $u'v'$ or the paths u-v and u'-v' are isomorphic in G, since they have same vanishing number. Hence G is isomorphic to the graph G'. □

§5. Characterization of Coloring Property

In this section, we give one characterization for 2-colorable and study about the relation between the coloring of ideal graph and the actual graph. Also, we find an upper bound for the chromatic number of a graph.

Theorem 5.1 *A graph G is 2-colorable if and only if $I_d(G)$ is 2-colorable.*

Proof It is obvious from the definition of ideal graph that a graph G has odd cycles if and only if the ideal graph $I_d(G)$ has odd cycles. We know that a graph G is 2-colorable if and only if it contains no odd cycles. Hence a graph G is 2-colorable if and only if $I_d(G)$ is 2-colorable.□

Theorem 5.2 *The strong vertices of a graph G can have the same colors in G and $I_d(G)$ under some 2-coloring if and only if all the edges of $I_d(G)$ have even vanishing number.*

Proof Assume that the strong vertices of a graph G have same colors in G and $I_d(G)$ under some 2-colorings. Let uv be an edge of $I_d(G)$. Then u and v are in different colors in $I_d(G)$ under a 2-coloring. If the vanishing number of uv is an odd number, then u and v have the same colors in G. Thus u or v differs by color in G from $I_d(G)$. This contradicts our assumption. Hence all edges of $I_d(G)$ have even vanishing number. Other part of this theorem is obvious.□

Theorem 5.3 *A graph G is k-colorable with $k \geq 3$ and the strong vertices of G can have the same colors as in $I_d(G)$ under a k-coloring if $I_d(G)$ is k-colorable.*

Proof Let $I_d(G)$ is k-colorable with $k \geq 3$. Assign the same colors for the strong vertices of G as in $I_d(G)$ under a k-coloring. Then for the weak vertices which are lying in the path of connecting strong vertices, we can use 3 colors such that G is k-colorable and the strong vertices of G can have the same colors as in $I_d(G)$. □

Corollary 5.4 *For any graph $G, \chi(G) \leq \chi(I_d(G)) \leq p_{id}$.*

Proof Proof follows from Theorem 5.3. □

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Math.Combin.Book Ser. Vol.3 (2011), 17-22

Pseudo-Smarandache Functions of First and Second Kind

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Abstract: In this paper we define two kinds of pseudo-Smarandache functions. We have investigated more than fifty terms of each pseudo-Smarandache function. We have proved some interesting results and properties of these functions.

Key Words: pseudo-Smarandache function, number, prime.

AMS(2010): 11P83

§1. Introduction

The pseudo-Smarandache function $Z(n)$ was introduced by Kashihara [4] as follows:

Definition 1.1 *For any integer $n \geq 1, Z(n)$ is the smallest positive integer m such that $1 + 2 + 3 + \ldots m$ is divisible by n.*

Alternately, $Z(n) = min\{m : m \in N : n \mid \dfrac{m(m+1)}{2}\}$.

The main results and properties of pseudo-Smarandache functions are available in [3]-[5]. We noticed that the sum $1 + 2 + 3 + \ldots + m$ can be replaced by the series of squares of first m natural numbers and the cubes of first m natural numbers respectively, to get the pseudo-Smarandache functions of first kind and second kind.

In the following we define pseudo-Smarandache functions of first kind and second kind.

Definition 1.2 *For any integer $n \geq 1$, the pseudo-Smarandache function of first kind, $Z_1(n)$ is the smallest positive integer m such that $1^2 + 2^2 + 3^2 \ldots + m^2$ is divisible by n.*

Alternately, $Z_1(n) = min\{m : m \in N : n \mid \dfrac{m(m+1)(2m+1)}{6}\}$.

Definition 1.3 *For any integer $n \geq 1$, the pseudo-Smarandache function of second kind, $Z_2(n)$ is the smallest positive integer m such that $1^3 + 2^3 + 3^3 \ldots + m^3$ is divisible by n.*

Alternately, $Z_2(n) = min\{m : m \in N : n \mid \dfrac{m^2(m+1)^2}{4}\}$.

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[1]Supported by UGC under the project No. 47-993/09 (WRO)

[2]Received April 12, 2011. Accepted August 15, 2011.

For ready reference we give below some values of $S(m)$s and $Z_1(n)$s, where $S(m)$ stands for the sum of the squares of first m natural numbers and $Z_1(n)$ stands for the pseudo-Smarandache function of first kind for the value n for $n \in N$.

Values of $S(m)$

$S(1) = 1$	$S(15) = 1240$	$S(29) = 8555$	$S(43) = 27434$
$S(2) = 5$	$S(16) = 1496$	$S(30) = 9455$	$S(44) = 29370$
$S(3) = 14$	$S(17) = 1785$	$S(31) = 10416$	$S(45) = 31395$
$S(4) = 30$	$S(18) = 2109$	$S(32) = 11440$	$S(46) = 33511$
$S(5) = 55$	$S(19) = 2470$	$S(33) = 12529$	$S(47) = 35726$
$S(6) = 91$	$S(20) = 2870$	$S(34) = 13685$	$S(48) = 38024$
$S(7) = 140$	$S(21) = 3311$	$S(35) = 14910$	$S(49) = 40425$
$S(8) = 204$	$S(22) = 3795$	$S(36) = 16206$	$S(50) = 42925$
$S(9) = 285$	$S(23) = 4324$	$S(37) = 17575$	$S(51) = 50882$
$S(10) = 385$	$S(24) = 4900$	$S(38) = 19019$	$S(52) = 48230$
$S(11) = 506$	$S(25) = 5525$	$S(39) = 20540$	$S(53) = 51039$
$S(12) = 650$	$S(26) = 6201$	$S(40) = 22140$	$S(54) = 53955$
$S(13) = 819$	$S(27) = 6930$	$S(41) = 23821$	$S(55) = 56980$
$S(14) = 1015$	$S(28) = 7714$	$S(42) = 25585$	$S(56) = 60116$

Values of $Z_1(n)$

$Z_1(1) = 1$	$Z_1(14) = 3$	$Z_1(27) = 40$	$Z_1(40) = 15$
$Z_1(2) = 3$	$Z_1(15) = 4$	$Z_1(28) = 7$	$Z_1(41) = 20$
$Z_1(3) = 4$	$Z_1(16) = 31$	$Z_1(29) = 14$	$Z_1(42) = 27$

$Z_1(4) = 7$	$Z_1(43) = 21$	$Z_1(17) = 8$	$Z_1(30) = 4$
$Z_1(5) = 2$	$Z_1(18) = 27$	$Z_1(31) = 15$	$Z_1(44) = 16$
$Z_1(6) = 4$	$Z_1(19) = 9$	$Z_1(32) = 63$	$Z_1(45) = 27$
$Z_1(7) = 3$	$Z_1(20) = 7$	$Z_1(33) = 22$	$Z_1(46) = 11$
$Z_1(8) = 15$	$Z_1(21) = 17$	$Z_1(34) = 8$	$Z_1(47) = 23$
$Z_1(9) = 13$	$Z_1(22) = 11$	$Z_1(35) = 7$	$Z_1(48) = 31$
$Z_1(10) = 4$	$Z_1(23) = 11$	$Z_1(36) = 40$	$Z_1(49) = 24$
$Z_1(11) = 5$	$Z_1(24) = 31$	$Z_1(37) = 18$	$Z_1(50) = 12$
$Z_1(12) = 8$	$Z_1(25) = 12$	$Z_1(38) = 19$	$Z_1(51) = 8$
$Z_1(13) = 6$	$Z_1(26) = 12$	$Z_1(39) = 13$	$Z_1(52) = 32$

§2. Some Results for Pseudo-Smarandache Functions of First Kind

Following results can be directly verified from the table given above.

(1) $Z_1(n) = 1$ only if $n = 1$.

(2) $Z_1(n) \geq 1$ for all $n \in N$.

(3) $Z_1(p) \leq p$, where p is a prime.

(4) If $Z_1(p) = n$, $p \neq 3$, then $p > n$.

Lemma 2.1 *If p is a prime then $Z_1(p) = p+1$, for $p = 2$ or 3. Also, $Z_1(p) = \dfrac{p-1}{2}$ for $p \geq 5$.*

Proof For $p = 2$ and 3, the verification is direct from the above table of $Z_1(n)$.

Let $S = 1^2 + 2^2 + 3^2 + \ldots + (\frac{p-1}{2})^2$. Then $S = \dfrac{p(p+1)(p-1)}{24}$. Hence p divides S. Also $p \nmid \dfrac{p-1}{2}$ as $\dfrac{p-1}{2} < p$. Let if possible (assumption) $p \mid 1^2 + 2^2 + \ldots + m^2$ where $m < \dfrac{p-1}{2}$. But in that case p divides every summand of the sum S. But $p \nmid (\frac{p-1}{2})^2$. Hence our assumption is wrong. Thus S is the minimum. Thus $Z_1(p) = \dfrac{p-1}{2}$ □

Lemma 2.2 *For $p = 2$, $Z_1(p^k) = p^{k+1} - 1$.*

Proof Straight verification confirms the result. □

Lemma 2.3 $Z_1(n) \geq max\{Z_1(N) : N \mid n\}$.

Proof Notice that in this case values of N are less than or equal to n and are divisors of n. Hence values of $Z_1(N)$ can not exceed $Z_1(n)$. □

Lemma 2.4 *Let* $n = \dfrac{k(k+1)(2k+1)}{6}$ *for some* $k \in N$, *then* $Z_1(n) = k$.

Proof The result is the immediate consequence of the fact that no previous value of $S(n)$ is divisible by k. □

Lemma 2.5 *It is not possible that* $Z_1(m) = m$ *for any* $m \in N$.

Proof Let if possible $Z_1(m) = m$. Then by definition m is the smallest of the positive integer which divides $1^2 + 2^2 + 3^2 + \ldots m^2$. Hence m does not divide $1^2 + 2^2 + 3^2 + \ldots (m-1)^2$. Let $1^2 + 2^2 + 3^2 + \ldots (m-1)^2 = k$. So, m divides $k + m^2$. Hence m divides k, a contradiction. □

Lemma 2.6 $S(m) = k$ *then* $S(m) = Z_1(2k+1)$.

Here $S(n)$ will stand for the sum of the cubes of first n natural numbers. Please find the table following.

Values of $S(n)$

$S(1) = 1$	$S(15) = 14400$	$S(29) = 189225$	$S(43) = 894916$
$S(2) = 9$	$S(16) = 18496$	$S(30) = 216225$	$S(44) = 980100$
$S(3) = 36$	$S(17) = 23409$	$S(31) = 246016$	$S(45) = 1071225$
$S(4) = 100$	$S(18) = 29241$	$S(32) = 278784$	$S(46) = 1168561$
$S(5) = 225$	$S(19) = 36100$	$S(33) = 314721$	$S(47) = 1272384$
$S(6) = 441$	$S(20) = 44100$	$S(34) = 354025$	$S(48) = 1382976$
$S(7) = 784$	$S(21) = 53361$	$S(35) = 396900$	$S(49) = 1500625$
$S(8) = 1296$	$S(22) = 64009$	$S(36) = 443556$	$S(50) = 1625625$
$S(9) = 2025$	$S(23) = 76176$	$S(37) = 494209$	
$S(10) = 3025$	$S(24) = 90000$	$S(38) = 549081$	

Values of $S(n)$(continue)

$S(11) = 4356$	$S(25) = 105625$	$S(39) = 608400$	
$S(12) = 6084$	$S(26) = 123201$	$S(40) = 672400$	
$S(13) = 8281$	$S(27) = 142884$	$S(41) = 741321$	
$S(14) = 11025$	$S(28) = 164836$	$S(42) = 815409$	

Values of $Z_2(n)$

$Z_2(1) = 1$	$Z_2(14) = 7$	$Z_2(27) = 8$	$Z_2(40) = 15$
$Z_2(2) = 3$	$Z_2(15) = 5$	$Z_2(28) = 7$	$Z_2(41) = 40$
$Z_2(3) = 2$	$Z_2(16) = 7$	$Z_2(29) = 28$	$Z_2(42) = 20$
$Z_2(4) = 3$	$Z_2(17) = 16$	$Z_2(30) = 15$	$Z_2(43) = 42$
$Z_2(5) = 4$	$Z_2(18) = 3$	$Z_2(31) = 30$	$Z_2(44) = 111$
$Z_2(6) = 3$	$Z_2(19) = 18$	$Z_2(32) = 15$	$Z_2(45) = 5$
$Z_2(7) = 6$	$Z_2(20) = 4$	$Z_2(33) = 11$	$Z_2(46) = 23$
$Z_2(8) = 7$	$Z_2(21) = 6$	$Z_2(34) = 16$	$Z_2(47) = 46$
$Z_2(9) = 2$	$Z_2(22) = 11$	$Z_2(35) = 14$	$Z_2(48) = 8$
$Z_2(10) = 4$	$Z_2(23) = 22$	$Z_2(36) = 3$	$Z_2(49) = 6$
$Z_2(11) = 10$	$Z_2(24) = 15$	$Z_2(37) = 36$	$Z_2(50) = 4$
$Z_2(12) = 3$	$Z_2(25) = 4$	$Z_2(38) = 19$	
$Z_2(13) = 12$	$Z_2(26) = 12$	$Z_2(39) = 12$	

§3. Some Results on Pseudo-Smarandache Function of Second Kind

Following properties are result of direct verification from the above tables.

(1) $Z_2(n) = n$ only for $n = 1$.

(2) $Z_2(p) = p - 1$, $p \neq 2$. $Z_2(p) = p + 1$ for $p = 2$.

(3) $Z_2(n) \geq max\{Z_2(N) : N \mid n\}$.

Following are some of the important results.

Lemma 3.1 *If $S(n) = k$ then $Z_2(k) = n$.*

Proof The proof follows from the definition of $Z_2(n)$. □

§4. Open Problem

Problem *What is the relation between $Z_1(n)$ and $Z_2(n)$?*

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Math.Combin.Book Ser. Vol.3 (2011), 23-33

On The Geometry of Hypersurfaces of a Certain Connection in a Quasi-Sasakian Manifold

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Abstract: The existence of new connection is proved. In particular case this connection reduces to several symmetric, semi-symmetric and quarter symmetric connections even some of them are not introduced so far . In this paper we define a geometry of hypersurfaces of a quarter symmetric semi metric connection in a quasi Sasakian manifold and consider its existence of Kahler structure, existence of a globally metric frame f-structure, integrability of distributions and geometry of their leaves with that connection.

Key Words: Hypersurfaces, quarter Symmetric semi-metric connection, quasi-sasakian manifold, Gauss and Weingarten equations.

AMS(2010): 53D12, 53C05

§1. Introduction

The concept of CR-submanifold of a Kahlerian manifold has been defined by A. Bejancu[1]. Later, A. Bejancu and N. Papaghiue [2], introduced and studied the notion of semi-invariant submanifold of a Sasakian manifold. Which are closely related to CR-submanifolds in a Kahlerian manifold. However the existence of the structure vector field implies some important changes.

The linear connection ∇ in an n-dimensional differentiable manifold M is called symmetric if its torsion tensor vanishes, otherwise it is non-symmetric.The connection ∇ is metric if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

In 1973, B. G. Schmidt [11] proved that if the holonomy group of ∇ is a subgroup of the orthogonal group $O(n)$, then ∇ is the Levi-Civita connection of a Riemannian metric. In 1924, A. Friedmann and J. A. Schouten [9] introduced the idea of a semi-symmetric linear connection in a differentiable manifold. A linear connection is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = u(Y)X - u(X)Y,$$

[1]Received February 26, 2011. Accepted August 16, 2011.

where u is a 1-form. A Hayden connection with the torsion tensor of the above form is a semi-symmetric metric connection. In 1970, K. Yano [13] considered a semi-symmetric metric connection and studied some of its properties. He proved that a Riemannian manifold is conformally flat if and only if it admits a semi-symmetric metric connection whose curvature tensor vanishes identically.He also proved that a Riemannian manifold is of constant curvature if and only if it admits a semi-symmetric connection for which the manifold is a group manifold, where a group manifold [8] is a differentiable manifold admitting a linear connection ∇ such that its curvature tensor R vanishes and its torsion tensor is covariantly constant with respect to ∇. In [12], L. Tamassy and T. Q. Binh proved that if in a Riemannian manifold of dimension ≥ 4, ∇ is a metric linear connection of non-vanishing constant curvature for which

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

then ∇ is the Levi-Civita connection.On the other hand, S. Golab [10] introduced the idea of a quarter symmetric linear connection if its torsion tensor T is of the form

$$T(X,Y) = u(Y)\phi X - u(X)\phi Y,$$

where u is a 1-form and ϕ is a tensor field of the type (1,1).

The purpose of the paper is to define and study quarter symmetric semi metric connection in a quasi-sasakian manifold and consider its Kahler structure, globally metric frame f-structure, integrability of distributions and geometry of their leaves. In Section 2, we recall some results and formulae for the later use. In Section 3, we prove the existence of a Kahler structure on and the existence of a globally metric frame f-structure in sence of S.I. Goldberg-K. Yano [6]. The Section 4, is concerned with integrability of distributions on and geometry of their leaves.

§2. Preliminaries

Let $\bar{M}$ be a real $2n+1$ dimensional differentiable manifold, endowed with an almost contact metric structure (f, ξ, η, g). Then we have from [4]

$$\begin{aligned}
&\text{(a)} \quad f^2 = -I + \eta \otimes \xi;\\
&\text{(b)} \quad \eta(\xi) = 1;\\
&\text{(c)} \quad \eta \circ f = 0;\\
&\text{(d)} \quad f(\xi) = 0;\\
&\text{(e)} \quad \eta(X) = g(X,\xi);\\
&\text{(f)} \quad g(fX, fY) = g(X,Y) - \eta(X)\eta(Y)
\end{aligned} \tag{2.1}$$

for any vector field X, Y tangent to $\bar{M}$, where I is the identity on the tangent bundle $\Gamma\bar{M}$ of $\bar{M}$. Throughout the paper, all manifolds and maps are differentiable of class C^∞.We denote by $F(\bar{M})$ the algebra of the differentiable functions on $\bar{M}$ and by $\Gamma(E)$ the $F(\bar{M})$ module of the sections of a vector bundle E over $\bar{M}$.

The Niyembuis tensor field, denoted by N_f, with respect to the tensor field f, is given by

$$N_f(X,Y) = [fX, fY] + f^2[X,Y] - f[fX,Y] + f[X,fY], \quad \forall X,Y \in \Gamma(T\bar{M})$$

And the fundamental 2-form Φ is given by

$$\Phi(X,Y) = g(X,fY) \quad \forall X,Y \in \Gamma(T\bar{M}). \tag{2.2}$$

The curvature tensor field of $\bar{M}$, denoted by $\bar{R}$ with respect to the Levi-Civita connection $\bar{\nabla}$,is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z \quad \forall X,Y,Z \in \Gamma(T\bar{M}). \tag{2.3}$$

(a) An almost contact metric manifold $\bar{M}$ (f, ξ, η, g) is called normal if

$$N_f(X,Y) + 2d\eta(X,Y)\xi = 0 \quad \forall X,Y \in \Gamma(T\bar{M}), \tag{2.4}$$

Or equivalently

$$(\bar{\nabla}_{fX} f)Y = f(\bar{\nabla}_X f)Y - g((\bar{\nabla}_X \xi, Y) \quad \forall X,Y \in \Gamma(T\bar{M}).$$

(b) The normal almost contact metric manifold $\bar{M}$ is called cosympletic if $d\Phi = d\eta = 0$

Let $\bar{M}$ be an almost contact metric manifold $\bar{M}$. According to [5] we say that $\bar{M}$ is a quasi-Sasakian manifold if and only if ξ is a Killing vector field and

$$(\bar{\nabla}_X f)Y = g(\bar{\nabla}_{fX}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{fX}\xi \quad \forall X,Y \in \Gamma(T\bar{M}). \tag{2.5}$$

Next we define a tensor field F of type $(1,1)$ by

$$FX = -\bar{\nabla}_X \xi \quad \forall X \in \Gamma(T\bar{M}). \tag{2.6}$$

A quarter symmetric semi metric connection ∇ on M is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(X) fY - g(fX,Y)\xi$$

$$(\bar{\nabla}_X f)Y = g(\bar{\nabla}_{fX}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{fX}\xi - g(X,Y)\xi + \eta(X)\eta(Y)\xi \quad \forall X,Y \in \Gamma(T\bar{M}), \tag{2.7}$$

$$\bar{\nabla}_X \xi = -FX \quad \forall X \in \Gamma(T\bar{M}). \tag{2.8}$$

From [5] we recall

Lemma 2.1 *Let $\bar{M}$ be a quasi-Sasakian manifold. Then we have*

$$\begin{aligned}
&\text{(a)} \quad (\bar{\nabla}_\xi f)X = 0 \quad \forall X \in \Gamma(T\bar{M});\\
&\text{(b)} \quad f \circ F = F \circ f \quad (c) \quad F\xi = 0\\
&\text{(d)} \quad g(FX,Y) + g(X,FY) = 0 \quad \forall X,Y \in \Gamma(T\bar{M});\\
&\text{(e)} \quad \eta \circ F = 0;\\
&\text{(f)} \quad (\bar{\nabla}_X F)Y = \bar{R}(\xi, X)Y \quad \forall X,Y \in \Gamma(T\bar{M}).
\end{aligned} \tag{2.9}$$

The tersor field f defined on $\bar{M}$ an f structure in sense of K. Yano that is

$$f^3 + f = 0.$$

Next let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold $\bar{M}$ and denote by N the unit vector field normal to M. Denote by the same symbol g the induced tensor metric on M, by ∇ the induced Levi-Civita connection on M and by $TM^{\perp}$ the normal vector bundle to M.The Gauss and Weingarten formulae of hypersurfaces of a quarter symmetric semi metric connections are

$$\begin{aligned} &\text{(a)} \quad \bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N; \\ &\text{(b)} \quad \bar{\nabla}_X N = -AX + \eta(X) fN, \end{aligned} \tag{2.10}$$

where A is the shape operator with respect to the section N. It is known that

$$B(X,Y) = g(AX,Y) \quad \forall X,Y \in \Gamma(TM). \tag{2.11}$$

Because the position of the structure vector field with respect to M is very important we prove the following results.

Theorem 2.1 *Let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold $\bar{M}$. If the structure vector field ξ is normal to M then $\bar{M}$ is cosympletic manifold and M is totally geodesic immersed in $\bar{M}$.*

Proof Because $\bar{M}$ is quasi-Sasakian manifold , then it is normal and $d\Phi = 0$ ([3]). By direct calculation using (2.10) (b), we infer

$$2d\eta(X,Y) = g(\bar{\nabla}_X\xi,Y) - g(\bar{\nabla}_Y\xi,X) = g(AY,X) - g(AX,Y) = 0 \quad \forall X,Y \in \Gamma(T\bar{M}). \tag{2.12}$$

From (2.10) (b) and (2.12) we deduce

$$0 = d\eta(X,Y) = g(Y,\bar{\nabla}_X\xi) = -g(AX,Y) = 0 \quad \forall X,Y \in \Gamma(T\bar{M}) \tag{2.13}$$

which proves that M is totally geodesic. From (2.13) we obtain $\bar{\nabla}_X\xi = 0 \quad \forall X \in \Gamma(T\bar{M})$ By using (2.8),(2.9)(b)and (2.1) (d) from the above relation we state

$$\bar{\nabla}_X\xi = -f\bar{\nabla}_{fX}\xi = 0 \quad \forall X \in \Gamma(T\bar{M}), \tag{2.14}$$

because $fX \in \Gamma(T\bar{M}) \quad \forall X \in \Gamma(T\bar{M})$. Using (2.14) and the fact that ξ is a Killing vector field, we deduce $d\eta = 0$ that is $\bar{M}$ is a cosympletic manifold. The proof is complete. □

Next we consider only the hypersurface which are tangent to ξ. Denote by $U = fN$ and from (2.1) (f), we deduce $g(U,U) = 1$. Moreover, it is easy to see that $U \in \Gamma(TM)$ .Denote by $D^{\perp} = Span(U)$ the 1-dimensional distribution generated by U, and by D the orthogonal complement of $D^{\perp} \oplus (\xi)$ in TM. It is easy to see that

$$fD = D, \quad D^{\perp} \subseteq TM^{\perp}; \quad TM = D \oplus D^{\perp} \oplus (\xi), \tag{2.15}$$

where $\oplus$ denote the orthogonal direct sum. According with [1] from (2.15) we deduce that M is a CR-submanifold of $\bar{M}$.

A CR-submanifold M of a quasi-Sasakian manifold $\bar{M}$ is called CR-product if both distributions $D \oplus (\xi)$ and $D^{\perp}$ are integrable and their leaves are totally geodesic submanifold of M.

Denote by P the projection morphism of TM to D and using the decomposion in (2.15) we deduce

$$X = PX + a(X)U + \eta(X)\xi \quad \forall X \in \Gamma(T\bar{M}), \tag{2.16}$$

$$fX = fPX + a(X)fU + \eta(fX)\xi \qquad \therefore \quad fX = fPX - a(X)fU,$$

Since

$$U = fN, fU = f^2N = -N + \eta(N)\xi = -N + g(N,\xi)\xi = -N,$$

where a is a 1-form on M defined by $a(X) = g(X,U), \quad X \in \Gamma(TM)$. From (2.16) using (2.1) (a) we infer

$$fX = tX - a(X)N \quad \forall X \in \Gamma(TM), \tag{2.17}$$

where t is a tensor field defined by $tX = fPX, \quad X \in \Gamma(TM)$

It is easy to see that

(a) $t\xi = 0$;
(b) $tU = 0$. (2.18)

§3. Induced Structures on a Hypersurface of a Quarter Symmetric Semi-Metric Connection in a Quasi-Sasakian Manifold

Let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold $\bar{M}$. From (2.1) (a), (2.17) and (2.18) we obtain $t^3 + t = 0$, that is the tensor field t defines an f-structure on M in sense of K.Yano [7]. Moreover, from (2.1) (a), (2.17), (2.18) we infer

$$t^2X = -X + a(X)U + \eta(X)\xi \quad \forall X \in \Gamma(TM). \tag{3.1}$$

Lemma 3.1 *On a hypersurface of a quarter symmetric semi metric connection M of a quasi-Sasakian manifold $\bar{M}$ the tensor field t satisfies:*

(a) $g(tX,tY) = g(X,Y) - \eta(X)\eta(Y) - a(X)a(Y)$, (3.2)
(b) $g(tX,Y) + g(X,tY) = 0 \quad \forall X,Y \in \Gamma(TM)$.

Proof From (2.1) (f), and (2.17) we deduce

$$g(X,Y) - \eta(X)\eta(Y) = g(fX,fY) = g(tX - a(X)N, tY - a(Y)N) = 0$$

$$= g(tX,tY) + a(X)a(Y)$$

$$g(tX,tY) = g(X,Y) - \eta(X)\eta(Y) - a(X)a(Y)$$

(*b*) $$g(tX,Y) + g(X,tY) = g(fX + a(X)N, Y) + g(X, fY + a(Y)N)$$

$$= g(fX,Y) + a(X)g(N,Y) + g(X,fY) + a(Y)g(X,N)$$

$$= g(fX, Y) + g(X, fY) = 0.$$

Lemma 3.2 *Let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold $\bar{M}$. Then*

(a) $FU = fA\xi + N;$
(b) $FN = A\xi - U;$
(c) $[U, \xi] = 0.$ (3.3)

Proof We take $X = U$, and $Y = \xi$ in (2.7) and obtain

$$f\bar{\nabla}_U\xi = -\bar{\nabla}_N\xi$$

Then using (2.1) (a), (2.8), (2.10)(b), we deduce the assertion (a). The assertion (b) follows from (2.1) (a), (2.9) (b) and (2.10) (b) and (3.3).Next by direct calculations, using (2.8), (2.9) (b) and (2.10) we derive

$$\bar{\nabla}_\xi U = (\bar{\nabla}_\xi f)N + f\bar{\nabla}_\xi N = -fA\xi = -FU = \bar{\nabla}_U\xi,$$

$$[U, \xi] = \bar{\nabla}_U\xi - \bar{\nabla}_\xi U = \bar{\nabla}_U\xi - \bar{\nabla}_U\xi = 0$$

Which prove assertion (c). By using the decomposition $T\bar{M} = TM \oplus TM^\perp$, we deduce

$$FX = \alpha X - \eta(AX)N, \quad \forall X \in \Gamma(T\bar{M}), \tag{3.4}$$

where α is a tensor field of type (1, 1) on M, since $g(FX, N) = -g(X, FN) = -g(X, A\xi - U) = -\eta(AX) + a(X) \quad \forall X \in \Gamma(T\bar{M})$. By using (2.7), (2.8), (2.10), (2.17) and (3.1), we obtain

Theorem 3.1 *Let M be a hypersurface of a quarter symmetric semi-metric connection in a quasi-sasakian manifold $\bar{M}$. Then the covariant derivative of a tensors t, a, η and α are given by*

(a) $(\nabla_X t)Y = g(FX, fY)\xi + \eta(Y)[\alpha tX - \eta(AX)U + a(X)U],$

$$-g(fX, fY)\xi - a(Y)AX + B(X, Y)U; \tag{3.5}$$

(b) $(\nabla_X a)Y = B(X, tY) + \eta(Y)\eta(AtX) - a(Y)\eta(X);$
(c) $(\nabla_X \eta)(Y) = g(Y, \nabla_X\xi) - \eta(Y)\eta(X);$
(d) $(\nabla_X \alpha)Y = (\nabla_X F)Y + B(X, Y)A\xi - B(X, Y)U - \eta(AY)AX \quad \forall X, Y \in \Gamma(TM),$

respectively, where R is the curvature tensor field of M.

From (2.7), (2.8), (2.18) (a), (b) and (3.5)(a) we get

Proposition 3.1 *On a hypersurface of a quarter symmetric semi metric connection M of a quasi-sasakian manifold $\bar{M}$, we have*

(a) $\nabla_X U = -tAX + \eta(AtX)\xi - a(tX)\xi;$ (3.6)
(b) $B(X, U) = a(AX) - \eta(X) \quad \forall X \in \Gamma(TM).$

Theorem 3.2 *Let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold $\bar{M}$. The tensor field t is a parallel with respect to the Levi Civita connection ∇ on M iff*

$$AX = [\eta(AX) - a(X)]\xi + [a(AX) - \eta(X)]U. \tag{3.7}$$

Proof Suppose that the tensor field t is parallel with respect to (∇), that is $\nabla t = 0$. By using (3.5) (a), we deduce

$$\eta(Y)[\alpha tX - \eta(AX)U + a(X)U] + g(FX, fY)\xi - g(fX, fY)\xi,$$

$$-a(Y)AX + B(X,Y)U = 0 \quad \forall X, Y \in \Gamma(TM), \tag{3.8}$$

Take $Y = U$ in (3.8) and using (2.10) (b), (2.11), (3.6) (b) we infer

$$\eta(U)[\alpha tX - \eta(AX)U + a(X)U] + g(FX, fU)\xi - g(fX, fU)\xi - a(U)AX + B(X,U)U = 0,$$

$$AX = [\eta(AX) - a(X)]\xi + [a(AX) - \eta(X)]U \quad \forall X, Y \in \Gamma(TM),$$

The proof is complete. □

Proposition 3.2 *Let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold $\bar{M}$. Then*

(a) $(\nabla_X a)Y = 0 \Leftrightarrow \nabla_X U = 0$ *and* $\eta(Y)a(tX) = a(Y)\eta(X)$;
(b) $(\nabla_X \eta)Y = 0 \Leftrightarrow \nabla_X \xi = 0$ *and* $\eta(X)\eta(Y) = 0 \quad \forall X, Y \in \Gamma(TM)$.

Proof Let $\forall X, Y \in \Gamma(TM)$ and using (2.11), (3.2) (b), (3.5) (b) and (3.6) (a) we obtain

$$\begin{aligned} g(\nabla_X U, Y) &= g(-tAX + \eta(AtX)\xi - a(tX)\xi, Y) \\ &= g(-tAX, Y) + \eta(AtX)g(\xi, Y) = a(tX)g(\xi, Y) \\ &= g(AX, tY) + \eta(AtX)\eta(Y) - \eta(Y)a(tX) + \eta(X)a(Y) - \eta(X)a(Y) \\ &= (\nabla_X a)Y - \eta(Y)a(tX) + \eta(X)a(Y). \end{aligned}$$

$$(\nabla_X a)Y = 0 \quad \Leftrightarrow \quad \nabla_X U = 0 \quad \text{and} \quad \eta(Y)a(tX) = \eta(X)a(Y),$$

$$(\nabla_X \eta)(Y) = g(Y, \nabla_X \xi) + \eta(X)\eta(Y),$$

$$(\nabla_X \eta)(Y) = 0 \quad \Leftrightarrow \quad \nabla_X \xi = 0 \quad \text{and} \quad \eta(X)\eta(Y) = 0.$$ □

According to Theorem 2 in [7], the tensor field

$$\bar{f} = t + \eta \otimes U - a \otimes \xi$$

defines an almost complex structure on M. Moreover, from Proposition 3.2 we deduce

Theorem 3.3 *Let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold $\bar{M}$. If the tensor fields t, a, η are parallel with respect to the connection ∇, then $\bar{f}$ defines a Kahler structure on M.*

§4. Integrability of Distributions on a Hypersurface of a Quarter Symmetric Semi-metric Connection in a Quasi-Sasakian Manifold $\bar{M}$

From Lemma 3.2 we obtain

Corollary 4.1 *On a hypersurface of a semi symmetric semi-metric connection M of a quasi-Sasakian manifold $\bar{M}$ there exists a 2-dimensional foliation determined by the integral distribution $D^{\perp} \oplus (\xi)$.*

Theorem 4.1 *Let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-Sasakian manifold $\bar{M}$. Then vskip 3mm*

(a) *A leaf of $D^{\perp} \oplus (\xi)$ is totally geodesic submanifold of M if and only if*

(1) $AU = a(AU)U + \eta(AU)\xi$ *and*

(2) $FN = a(FN)U.$ (4.1)

(b) *A leaf of $D^{\perp} \oplus (\xi)$ is totally geodesic submanifold of $\bar{M}$ if and only if*

(1) $AU = 0$ *and*

(2) $a(FX) = a(FN) = 0, \forall X \in \Gamma(D)$.

Proof (a) Let M^* be a leaf of integrable distribution $D^{\perp} \oplus (\xi)$ and h^* be the second fundamental form of the immersion $M^* \to M$. By using (2.1) (f), and (2.10) (b) we get

$$\begin{aligned} g(h^*(U,U),X) &= g(\bar{\nabla}_U U, X) = -g(N, (\bar{\nabla}_U f)X - g((\bar{\nabla}_U N, fX), \\ &= 0 - g(-AU - \eta(U)N, fX) = g(AU, fX) + \eta(U)g(N, fX), \\ &= g(AU, fX) \quad \forall X \in \Gamma(TM) \end{aligned} \tag{4.2}$$

and

$$g(h^*(U,\xi),X) = g(\bar{\nabla}_U \xi, X) = -g(FU, X) = g(FN, fX) \quad \forall X \in \Gamma(TM), \tag{4.3}$$

Because $g(FU,N) = 0$ and $f\xi = 0$ the assertion (a) follows from (4.2) and (4.3).

(b) Let h_1 be the second fundamental form of the immersion $M^* \to M$. It is easy to see that

$$h_1(X,Y) = h^*(X,Y) + B(X,Y)N, \quad \forall X, Y \in \Gamma(D^{\perp} \oplus (\xi)). \tag{4.4}$$

From (2.8) and (2.11) we deduce

$$g(h_1(U,U),N) = g(\bar{\nabla}_U U, N) = a(AU), \tag{4.5}$$

$$g(h_1(U,\xi),N) = g(\bar{\nabla}_U \xi, N) = a(FU), \tag{4.6}$$

The assertion (b) follows from (4.3)-(4.6).

Theorem 4.2 *Let M be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold $\bar{M}$. Then*

(a) *the distribution* $D \oplus (\xi)$ *is integrable iff* $g(AfX + fAX, Y) = 0, \quad \forall X, Y \in \Gamma(D)$;(4.7)

(b) *the distribution* D *is integrable iff (4.7) holds and*

$$FX = \eta(AtX)U - \eta(AX)N, \quad (equivalent with FD \perp D) \quad \forall X \in \Gamma(D)$$

(c) The distribution $D \oplus D^{\perp}$ *is integrable iff* $FX = 0, \quad \forall X \in \Gamma(D)$.

Proof Let $X, Y \in \Gamma(D)$. Since ∇ is a torsion free and ξ is a Killing vector field, we infer

$$g([X,\xi],U) = g(\bar{\nabla}_X \xi, U) - g(\bar{\nabla}_\xi X, U) = g(\nabla_X \xi, U) + g(\nabla_U \xi, X) = 0 \quad \forall X \in \Gamma(D) \tag{4.8}$$

Using (2.1) (a), (2.10) (a) we deduce

$$\begin{aligned} g([X,Y],U) &= g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, U) = g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, fN) \\ &= g(\bar{\nabla}_Y fX - \bar{\nabla}_X fY, N) = g(AfX + fAX, Y) \quad \forall X, Y \in \Gamma(D). \end{aligned} \tag{4.9}$$

Next by using (2.8) (2.9) (d) and the fact that ∇ is a metric connection we get

$$g([X,Y],\xi) = g(\bar{\nabla}_X Y, \xi) - g(\bar{\nabla}_Y X, \xi) = 2g(FX, Y) \quad \forall X, Y \in \Gamma(D). \tag{4.10}$$

The assertion (a) follows from (4.8), (4.9) and assertion (b) follows from (4.8)-(4.10). Using (2.8) and (2.9) we obtain

$$g([X,U],\xi) = g(\bar{\nabla}_X U, \xi) - g(\bar{\nabla}_U X, \xi) = 2g(FX, U) \quad \forall X \in \Gamma(D) \tag{4.11}$$

Taking on account of

$$g(FX, N) = g(FfX, fN) = g(FfX, U) \quad \forall X \in \Gamma(D). \tag{4.12}$$

The assertion (c) follows from (4.10) and (4.11). □

Theorem 4.3 *Let* M *be a hypersurface of a quarter symmetric semi metric connection in a quasi-sasakian manifold* $\bar{M}$. *Then*

(a) *the distribution* D *is integrable and its leaves are totally geodesic immersed in* M *if and only if*

$$FD \perp D \quad and \quad AX = a(AX)U + \eta(AX)\xi, \quad \forall X \in \Gamma(D); \tag{4.13}$$

(b) *the distribution* $D \oplus (\xi)$ *is integrable and its leaves are totally geodesic immersed in if and only if*

$$AX = a(AX)U, \quad X \in \Gamma(D) \quad and \quad FU = 0; \tag{4.14}$$

(c) *the distribution* $D \oplus D^{\perp}$ *is integrable and its leaves are totally geodesic immersed in* M *if and only if .*

$$FX = 0, \quad X \in \Gamma(D).$$

Proof Let M_1^* be a leaf of integrable distribution D and h_1^* the second fundamental form of immersion $M_1^* \to M$. Then by direct calculation we infer

$$g(h_1^*(X,Y),U) = g(\bar{\nabla}_X Y, U) = -g(Y, \nabla_X U) = -g(AX, tY), \tag{4.15}$$

and

$$g(h_1^*(X,Y),\xi) = g(\bar{\nabla}_X Y,\xi) = g(FX,Y) \quad \forall X,Y \in \Gamma(D). \tag{4.16}$$

Now suppose M_1^* is a totally submanifold of M. Then (4.13) follows from (4.15) and (4.16). Conversely suppose that (4.13) is true. Then using the assertion (b) in Theorem 4.2 it is easy to see that the distribution D is integrable. Next the proof follows by using (4.15) and (4.16). Next, suppose that the distribution $D \oplus (\xi)$ is integrable and its leaves are totally geodesic submanifolds of M. Let M_1 be a leaf of $D \oplus (\xi)$ and h_1 the second fundamental form of immersion $M_1 \to M$. By direct calculations, using (2.8), (2.10) (b), (3.2) (b) and (3.6) (c), we deduce

$$g(h_1(X,Y),U) = g(\bar{\nabla}_X Y,U) = -g(AX,tY), \quad \forall X,Y \in \Gamma(D) \tag{4.17}$$

and

$$g(h_1(X,\xi),U) = g(\bar{\nabla}_X \xi,U) = -g(FU,X), \quad \forall X \in \Gamma(D). \tag{4.18}$$

Then the assertion (b) follows from (4.12), (4.17), (4.18) and the assertion (a) of Theorem 4.2 . Next let $\bar{M}_1$ a leaf of the integrable distribution $D \oplus D^{\perp}$ and $\bar{h_1}$ the second fundamental form of the immersion $M_1 \to M$. By direct calculation we get

$$g(\bar{h_1}(X,Y),\xi) = g(FX,Y), \quad \forall X \in \Gamma(D), Y \in \Gamma(D \oplus D^{\perp}). \tag{4.19}$$

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Math.Combin.Book Ser. Vol.3 (2011), 34-46

Complementary Signed Domination Number of Certain Graphs

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Abstract: Let $G=(V,E)$ be a simple graph, $k\geq 1$ an integer and let $f:V(G)\rightarrow\{-k,k-1,\cdots,-1,1,\cdots,k-1,k\}$ be $2k$ valued function. If $\sum\limits_{x\in N(v)} f(x)\geq k$ for each $v\in V(G)$, where $N(v)$ is the open neighborhood of v, then f is a Smarandachely complementary k-signed dominating function on G. The weight of f is defined as $w(f)=\sum_{v\in V} f(v)$ and the Smarandachely complementary k-signed domination number of G is defined as $\gamma^S_{cs}(G)=\min\{w(f): f$ is a minimal complementary signed dominating function of $G\}$. Particularly, a Smarandachely complementary 1-signed dominating function or family is called a complementary singed dominating function or family on G with abbreviated notation $\gamma_{cs}(G)$, the Smarandachely complementary 1-signed domination number of G. In this paper, we determine the value of complementary signed domination number for some special class of graphs. We also determine bounds for this parameter and exhibit the sharpness of the bounds. We also characterize graphs attaining the bounds in some special classes.

Key Words: Smarandachely complementary k-signed dominating function, Smarandachely complementary k-signed dominating number, dominating function, signed dominating function, complementary signed dominating function.

AMS(2010): 05C69

§1. Introduction

By a graph we mean a finite, undirected connected graph without loops or multiple edges. Terms not defined here are used in the sense of Haynes et. al. [3] and Harary [2].

Let $G=(V,E)$ be a graph with n vertices and m edges. A subset $S\subseteq V$ is called a dominating set of G if every vertex in V-S is adjacent to at least one vertex in S.

A function $f:V\rightarrow\{0,1\}$ is called a dominating function of G if $\sum_{u\in N[v]} f(u)\geq 1$ for every $v\in V$. Dominating function is a natural generalization of dominating set. If S is a dominating set, then the characteristic function is a dominating function.

Generally, let $f:V(G)\rightarrow\{-k,k-1,\cdots,-1,1,\cdots,k-1,k\}$ be $2k$ valued function. If $\sum\limits_{x\in N(v)} f(x)\geq k$ for each $v\in V(G)$, where $N(v)$ is the open neighborhood of v, then f is a *Smarandachely complementary k-signed dominating function* on G. The weight of f is defined

[1]Received February 12, 2011. Accepted August 18, 2011.

as $w(f) = \sum_{v\in V} f(v)$ and the *Smarandachely complementary k-signed domination number* of G is defined as $\gamma_{cs}^S(G) = \min\{w(f) : f$ is a minimal complementary signed dominating function of $G\}$. Particularly, if $k = 1$, a Smarandachely complementary 1-signed dominating function is a function $f : V \to \{+1, -1\}$ such that $\sum_{u\in N[v]} f(u) \geq 1$ for every $v \in V$ on G with abbreviated notation $\gamma_{cs}^S(G) = \gamma_{cs}(G) = \min\{w(f) : f$ is a minimal complementary signed dominating function of $G\}$, the Smarandachely complementary 1-signed domination number of G. Signed dominating function is defined in [1].

Definition 1.1 *A caterpillar is a tree T for which removal of all pendent vertices leaves a path.*

Definition 1.2 *The wheel W_n is defined to be the graph $K_1 + C_{n-1}$ for $n \geq 4$.*

§2. Main Results

Definition 2.1 *A function $f : V \to \{+1, -1\}$ is called a complementary signed dominating function of G if $\sum_{u\notin N[v]} f(u) \geq 1$ for every $v \in V$ with $deg(v) \neq n-1$. The weight of a complementary signed dominating function f is defined as $w(f) = \sum_{v\in V} f(v)$.*

The complementary signed domination number of G is defined as

$$\gamma_{cs}(G) = \min\{w(f) : \ f \ is \ a \ minimal \ complementary \ signed \ dominating \ function \ of \ G\}.$$

Example 2.2 Consider the graph G given in Fig 2.1

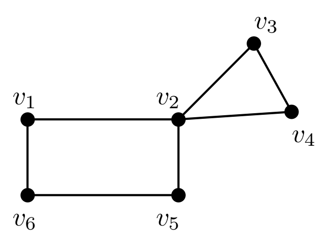

Fig.2.1

Define $f : V(G) \to \{+1, -1\}$ by $f(v_1) = f(v_3) = f(v_4) = f(v_6) = 1$ and $f(v_2) = f(v_5) = -1$. It is easy to observe that f is a minimal complementary signed dominating function with minimum weight and so $\gamma_{cs}(G) = 2$.

Theorem 2.3 *Let T_n be a caterpillar on $2n$ vertices obtained from a path $v_1, v_2, \ldots, v_n$ on n vertices by adding n new vertices $u_1, u_2, \ldots, u_n$ and joining u_i to v_i with an edge for each i. Then $\gamma_{cs}(T_n) = 4$.*

Proof The proof is divided into cases following.

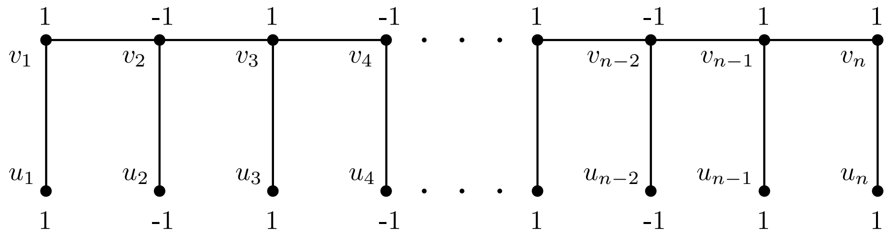

Fig.2.2

Case i n is even.

Define $f : V(T_n) \to \{+1, -1\}$ as follows :

$$f(v_i) = f(u_i) = \begin{cases} +1 & \text{if } 1 \leq \text{i} < \text{n and i is odd,} \\ -1 & \text{if } 2 \leq \text{i} < \text{n and i is even.} \end{cases}$$

$f(v_n) = f(u_n) = +1$. We claim that f is a complementary signed dominating function.

For odd i with $1 \leq i < n$,

$$\sum_{w \notin N[u_i]} f(w) = -(n-2) + [(n-2) - 2] + 4 = 2.$$

Also,

$$\sum_{w \notin N[u_n]} f(w) = -(n-2) + [(n-2) - 2] + 4 = 2.$$

For even i with $2 \leq i < n$,

$$\sum_{w \notin N[u_i]} f(w) = -[(n-2) - 2] + (n-2) + 4 = 6.$$

For $2 \leq i < n - 2$,

$$\sum_{w \notin N[v_i]} f(w) = -[(n-2) - 2] + (n-2) - 2 + 4 = 4,$$

$$\sum_{w \notin N[v_1]} f(w) = -[(n-2) - 1] + [(n-2) - 2] + 4 = 3,$$

$$\sum_{w \notin N[v_{n-1}]} f(w) = -[(n-2) - 1] + (n-2) + 4 - 3 = 2,$$

$$\sum_{w \notin N[v_{n-2}]} f(w) = -[(n-2) - 2] + (n-2) - 1 + 4 - 1 = 4,$$

$$\sum_{w \notin N[v_n]} f(w) = -(n-2) + (n-2) + 4 - 3 = 1.$$

Therefore f is a complementary signed dominating function. Since $\sum_{w \notin N[v_n]} f(w) = 1$, the labeling is minimum with respect to the vertices $v_1, v_2, \ldots, v_{n-2}$ and $u_1, u_2, \ldots, u_{n-1}$.

If u_{n-1} is given value -1, then $\sum_{u\notin N[u_n]} f(u) = 0$. It is easy to observe that $\sum_{v\in V[T_n]} f(v) = 4$ is minimum for this particular complementary signed dominating function. Hence $\gamma_{cs}(T_n) = 4$ if n is even.

Case ii n is odd.

Define $f : V(T_n) \to \{+1, -1\}$ as follows :

$$f(v_i) = \begin{cases} +1 & \text{for } 1 \le \mathrm{i} \le \mathrm{n} - 2 \text{ and i is odd,} \\ -1 & \text{for } 2 \le \mathrm{i} \le \mathrm{n} - 3 \text{ and i is even.} \end{cases}$$

and $f(v_{n-1}) = f(v_n) = +1$.

$$f(u_i) = \begin{cases} +1 & \text{if } 1 \le \mathrm{i} \le \mathrm{n} \text{ and i is odd,} \\ -1 & \text{if } 2 \le \mathrm{i} \le \mathrm{n} - 1 \text{ and i is even.} \end{cases}$$

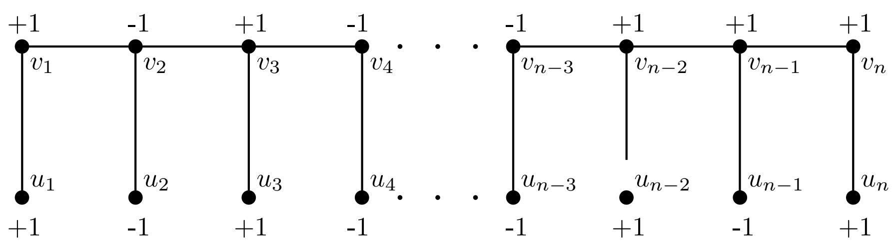

Fig.2.3

We claim that f is a complementary signed dominating function.

For odd i with $1 \le i \le n-4$,

$$\sum_{w\notin N[u_i]} f(w) = -[(n-3)+1] + [(n-3)+1-2] + 4 = 2.$$

For even i with $2 \le i \le n-3$,

$$\sum_{w\notin N[u_i]} f(w) = -[(n-3)+1-2] + (n-3)+1+4 = 6.$$

Also

$$\sum_{w\notin N[u_{n-2}]} f(w) = -[(n-3)+1] + (n-3)+1+4-2 = 2,$$

$$\sum_{w\notin N[u_{n-1}]} f(w) = -[(n-3)+1-1] + [(n-3)+1-1] + 4 = 4$$

and

$$\sum_{w\notin N[u_n]} f(w) = -[(n-3)+1] + [(n-3)+1] + 4 - 2 = 2.$$

For $2 \le i \le n-4$,

$$\sum_{w \notin N[v_i]} f(w) = -[(n-3)+1-2] + [(n-3)+1-2] + 4 = 4,$$
$$\sum_{w \notin N[v_1]} f(w) = -[(n-3)+1-1] + [(n-3)+1-2] + 4 = 3,$$
$$\sum_{w \notin N[v_{n-3}]} f(w) = -[(n-3)+1-2] + [(n-3)+1-1] + 4 - 1 = 4,$$
$$\sum_{w \notin N[v_{n-2}]} f(w) = -[(n-3)+1-1] + [(n-3)+1-1] + 4 - 2 = 2,$$
$$\sum_{w \notin N[v_{n-1}]} f(w) = -[(n-3)+1-1] + [(n-3)+1-1] + 4 - 2 = 2,$$
$$\sum_{w \notin N[v_n]} f(w) = -[(n-3)+1] + [(n-3)+1-1] + 4 - 2 = 1.$$

Therefore f is a complementary signed dominating function. Since $\sum_{w \notin N[v_n]} f(w) = 1$, the labeling is minimum with respect to the vertices $v_1, v_2, \ldots, v_{n-2}$ and $u_1, u_2, \ldots, u_{n-1}$.

If u_{n-2} is given value -1, then $\sum_{w \notin N[v_{n-1}]} f(w) = 0$. It is easy to observe that $\sum_{v \in V[T_n]} f(v) = 4$ is minimum for this particular complementary signed dominating function. Hence $\gamma_{cs}(T_n) = 4$ if n is odd. Therefore $\gamma_{cs}(T_n) = 4$ for all n. □

Theorem 2.4 *Let P_n be a path on n vertices and each vertex of P_n is a support which is adjacent to exactly two pendent vertices. Such a graph is called a caterpillar and denoted by T. Then*

$$\gamma_{cs}(T) = \begin{cases} 3 & \text{if } n \text{ is odd, } n \ge 3 \\ 4 & \text{if } n \text{ is even, } n \ge 4. \end{cases}$$

Proof Let the vertices of the path P_n be $v_1, v_2, \ldots, v_n$ and let each vertex v_i be adjacent to exactly two pendent vertices namely u_i and w_i.

Case i n is odd.

Define $f : V(T) \to \{+1, -1\}$ as follows :

$$f(v_i) = \begin{cases} +1 & \text{if } i \text{ is odd,} \\ -1 & \text{if } i \text{ is even.} \end{cases}$$

$f(u_i) = +1$ for all i and

$$f(w_i) = \begin{cases} +1 & \text{if } i = 1, \\ -1 & \text{if } 2 \le i \le n. \end{cases}$$

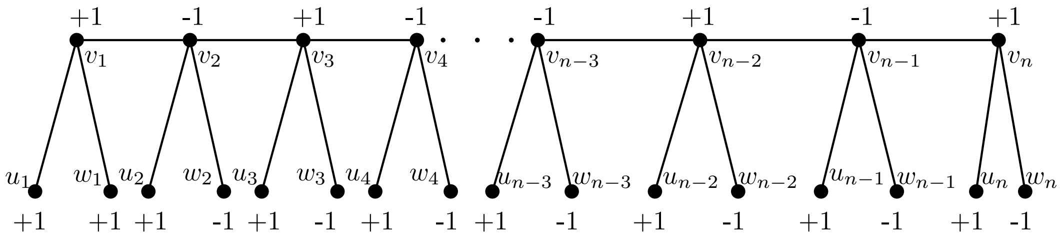

Fig.2.4

We claim that f is a complementary signed dominating function. We have,

$$\sum_{w \notin N[u_1]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left(\frac{n-1}{2}\right) + 2 - 1 + (n-1) - (n-1) = 1.$$

For even i with $2 \leq i \leq n-1$,

$$\sum_{w \notin N[u_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - \left[\left(\frac{n-1}{2}\right) - 1\right] + 2 + (n-1) - 1 - (n-1) = 3.$$

For odd i with $3 \leq i \leq n$,

$$\sum_{w \notin N[u_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left(\frac{n-1}{2}\right) + 2 + (n-1) - 1 - (n-1) = 1,$$

$$\sum_{w \notin N[w_1]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left(\frac{n-1}{2}\right) + 2 - 1 + (n-1) - (n-1) = 1,$$

For even i with $2 \leq i \leq n-1$,

$$\sum_{w \notin N[w_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - \left[\left(\frac{n-1}{2}\right) - 1\right] + 2 + (n-1) - [(n-1) - 1] = 5.$$

For odd i with $3 \leq i \leq n$,

$$\sum_{w \notin N[w_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left(\frac{n-1}{2}\right) + 2 + (n-1) - [(n-1) - 1] = 3,$$

$$\sum_{w \notin N[v_1]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left[-1 + \left(\frac{n-1}{2}\right)\right] + 2 - 2 + (n-1) - (n-1) = 1.$$

For even i with $2 \leq i \leq n-1$,

$$\sum_{w \notin N[v_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 2 - \left[\left(\frac{n-1}{2}\right) - 1\right] + 2 + (n-1) - 1 - [(n-1) - 1] = 2.$$

For odd i with $3 \leq i < n$,

$$\sum_{w \notin N[v_i]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left[\frac{n-1}{2} - 2\right] + 2 + (n-1) - 1 - [(n-1) - 1] = 4,$$

$$\sum_{w \notin N[v_n]} f(w) = \left(\frac{n-1}{2}\right) + 1 - 1 - \left[\left(\frac{n-1}{2}\right) - 1\right] + 2 + (n-1) - 1 - [(n-1) - 1] = 3.$$

Therefore f is a complementary signed dominating function. Since $\sum_{w\notin N[w_1]} f(w) = 1$, the labeling is minimum with respect to the vertices $v_2, v_3, \ldots, v_n$ and $w_2, w_3, \ldots, w_n$, $u_1, u_2, \ldots, u_n$.

If u_1 is given value -1, then $\sum_{w\notin N[v_i]} f(w) = 0$ for even i with $2 \leq i \leq n-1$. It is easy to observe that $\sum_{v\in V(T)} f(v) = 3$ is minimum for this particular complementary signed dominating function. Therefore $\gamma_{cs}(T) = 3$ if n is odd and $n \geq 3$.

Case ii n is even.

Define $f : V(T) \to \{+1, -1\}$ as follows :

$$f(v_i) = \begin{cases} +1 & \text{for } 1 \leq i \leq 4 \text{ and } 5 \leq i \leq n, i \text{ is odd,} \\ -1 & \text{for } 6 \leq i \leq n \text{and } i \text{ is even.} \end{cases}$$

$f(u_i) = +1$ for $1 \leq i \leq n$ and $f(w_i) = -1$ for $1 \leq i \leq n$

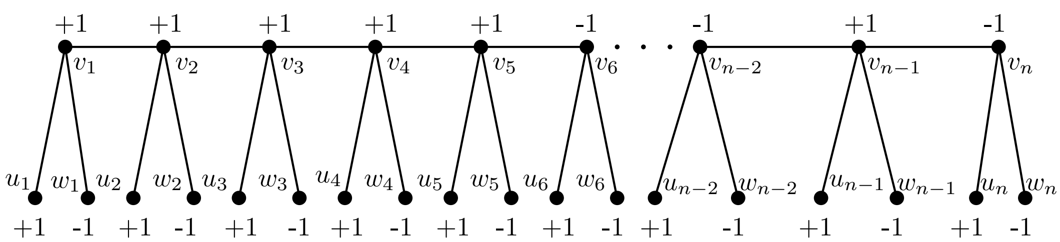

Fig.2.5

We claim that f is a complementary signed dominating function.

$$\sum_{w\notin N[v_1]} f(w) = 4 - 2 + \left(\frac{n-4}{2}\right) - \left(\frac{n-4}{2}\right) + (n-1) - (n-1) = 2.$$

For $i = 2,\ 3$,

$$\sum_{w\notin N[v_i]} f(w) = 4 - 3 + \left(\frac{n-4}{2}\right) - \left(\frac{n-4}{2}\right) + (n-1) - (n-1) = 1,$$

$$\sum_{w\notin N[v_4]} f(w) = 4 - 2 + \left(\frac{n-4}{2}\right) - 1 - \left(\frac{n-4}{2}\right) + (n-1) - (n-1) = 1,$$

$$\sum_{w\notin N[v_5]} f(w) = 4 - 1 + \left(\frac{n-4}{2}\right) - 1 - \left[\left(\frac{n-4}{2}\right) - 1\right] + (n-1) - (n-1) = 3.$$

For odd i with $7 \leq i \leq n-1$,

$$\sum_{w\notin N[v_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 1 - \left[\frac{n-4}{2} - 2\right] + (n-1) - (n-1) = 5.$$

For even i with $6 \leq i < n$,

$$\sum_{w\notin N[v_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 2 - \left[\left(\frac{n-4}{2}\right) - 1\right] + (n-1) - (n-1) = 3,$$

$$\sum_{w\notin N[v_n]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 1 - \left[\left(\frac{n-4}{2}\right) - 1\right] + (n-1) - (n-1) = 4.$$

For $1 \leq i \leq 4$,

$$\sum_{w \notin N[u_i]} f(w) = 4 - 1 + \left(\frac{n-4}{2}\right) - \left(\frac{n-4}{2}\right) + (n-1) - n = 2.$$

For odd i with $5 \leq i \leq n-1$,

$$\sum_{w \notin N[u_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 1 - \left(\frac{n-4}{2}\right) + (n-1) - n = 2.$$

For even i with $6 \leq i \leq n$,

$$\sum_{w \notin N[u_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - \left[\frac{n-4}{2} - 1\right] + (n-1) - n = 4.$$

For $1 \leq i \leq 4$,

$$\sum_{w \notin N[w_i]} f(w) = (4-1) + \frac{n-4}{2} - \left(\frac{n-4}{2}\right) + n - (n-1) = 4.$$

For odd i with $5 \leq i \leq n-1$,

$$\sum_{w \notin N[w_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - 1 - \left(\frac{n-4}{2}\right) + n - [n-1] = 4.$$

For even i with $6 \leq i \leq n$,

$$\sum_{w \notin N[w_i]} f(w) = 4 + \left(\frac{n-4}{2}\right) - \left[\left(\frac{n-4}{2}\right) - 1\right] + n - (n-1) = 6.$$

Therefore f is a complementary signed dominating function. Since $\sum_{w \notin N[v_2]} f(w) = 1$, the labeling is minimum with respect to the vertices $v_4, v_5, \ldots, v_n$, $u_1, u_3, \ldots, u_n$ and $w_1, w_3, \ldots, w_n$.

If v_4 is given value -1, then $\sum_{w \notin N[v_1]} f(w) = 0$. It is easy to observe that $\sum_{v \in V(T)} f(v) = 4$ is minimum for this particular complementary signed dominating function. Therefore $\gamma_{cs}(T) = 4$ if n is even and $n \geq 4$. $\square$

Theorem 2.5 *For a bipartite graph $K_{m,n}$,*

$$\gamma_{cs}(K_{m,n}) = \begin{cases} 5 & \textit{if exactly one of } m, n \textit{ is odd,} \\ 6 & \textit{if both } m \textit{ and } n \textit{ are odd,} \\ 4 & \textit{if both } m \textit{ and } n \textit{ are even,} \end{cases}$$

where $2 \leq m \leq n$.

Proof Let (V_1, V_2) be the partition of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Let the vertices of V_1 be $v_1, v_2, \ldots, v_m$ and let the vertices of V_2 be $u_1, u_2, \ldots, u_n$. Define $f : V(K_{m,n}) \to \{+1, -1\}$ as follows :

$$f(v_i) = \begin{cases} -1 & \text{if} \quad 1 \leq i \leq \dfrac{m-2}{2}, \\ +1 & \text{if} \quad \dfrac{m-2}{2} < i \leq m, \end{cases}$$

when m is even and

$$f(v_i) = \begin{cases} -1 & \text{if } \ 1 \leq i \leq \dfrac{m-3}{2}, \\ +1 & \text{if } \ \dfrac{m-3}{2} < i \leq m, \end{cases}$$

when m is odd

$$f(u_i) = \begin{cases} -1 & \text{if } \ 1 \leq i \leq \dfrac{n-2}{2}, \\ +1 & \text{if } \ \dfrac{n-2}{2} < i \leq n, \end{cases}$$

when n is even and

$$f(u_i) = \begin{cases} -1 & \text{if } \ 1 \leq i \leq \dfrac{n-3}{2}, \\ +1 & \text{if } \ \dfrac{n-3}{2} < i \leq n, \end{cases}$$

when n is odd.

Case i m is even.

Let v_i be a vertex with $f(v_i) = -1$. Then

$$\begin{aligned} \sum_{u \notin N[v_i]} f(u) &= (-1)\left[\frac{m-2}{2} - 1\right] + m - \left(\frac{m-2}{2}\right) \\ &= -(m-2) + 1 + m = 3 \end{aligned}$$

Let v_i be a vertex with $f(v_i) = +1$. Then

$$\begin{aligned} \sum_{u \notin N[v_i]} f(u) &= (-1)\left(\frac{m-2}{2}\right) + m - \left(\frac{m-2}{2}\right) - 1 \\ &= -(m-2) + m - 1 = 1 \end{aligned}$$

Case ii m is odd.

Let v_i be a vertex with $f(v_i) = -1$. Then

$$\begin{aligned} \sum_{u \notin N[v_i]} f(u) &= (-1)\left[\left(\frac{m-3}{2}\right) - 1\right] + m - \left(\frac{m-3}{2}\right) \\ &= -(m-3) + 1 + m = 4 \end{aligned}$$

Let v_i be a vertex with $f(v_i) = +1$. Then

$$\sum_{u \notin N[v_i]} f(u) = (-1)\left(\frac{m-3}{2}\right) + m - \left(\frac{m-3}{2}\right) - 1 = 2$$

Case iii n is even.

The proof is similar to case (i) replacing m and v_i by n and u_i.

Case iv n is odd.

The proof is similar to case (ii) replacing m and v_i by n and u_i.

If the number of vertices with function -1 is increased by 1, a vertex with function value $+1$ will not satisfy the condition necessary for a complementary signed dominating function. Therefore f is a complementary signed dominating function.

Case I Exactly one of m or n is odd.

When m is even and n is odd, then

$$\begin{aligned}\gamma_{cs}(K_{m,n}) &= \sum_{v\in V(K_{m,n})} f(v)\\ &= (-1)\left(\frac{m-2}{2}\right)+m-\left(\frac{m-2}{2}\right)+(-1)\left(\frac{n-3}{2}\right)+n-\left(\frac{n-3}{2}\right)\\ &= -(m-2)+m-(n-3)+n=5\end{aligned}$$

When m is odd and n is even

$$\begin{aligned}\gamma_{cs}(K_{m,n}) &= \sum_{v\in V(K_{m,n})} f(v)\\ &= -\left(\frac{m-3}{2}\right)+m-\left(\frac{m-3}{2}\right)-\left(\frac{n-2}{2}\right)+n-\left(\frac{n-2}{2}\right)\\ &= -(m-3)+m-(n-2)+n=5\end{aligned}$$

Case II Both m and n are even.

$$\begin{aligned}\gamma_{cs}(K_{m,n}) &= -\left(\frac{m-2}{2}\right)+m-\left(\frac{m-2}{2}\right)-\left(\frac{n-2}{2}\right)+n-\left(\frac{n-2}{2}\right)\\ &= -(m-2)+m-(n-2)+n=4\end{aligned}$$

Case III Both m and n are odd.

$$\begin{aligned}\gamma_{cs}(K_{m,n}) &= -\left(\frac{m-3}{2}\right)+m-\left(\frac{m-3}{2}\right)-\left(\frac{n-3}{2}\right)+n-\left(\frac{n-3}{2}\right)\\ &= -(m-3)+m-(n-3)+n=6\end{aligned}$$

□

Remark 2.6 $\gamma_{cs}(K_{m,n})=\gamma_s(K_{m,n})$ for $m,n>3$.

We observe that $\gamma_{cs}(W_5)=3$, $\gamma_{cs}(W_6)=4$, $\gamma_{cs}(W_7)=1$, $\gamma_{cs}(W_8)=4$, $\gamma_{cs}(W_9)=3$ and $\gamma_{cs}(W_{10})=2$. We determine $\gamma_{cs}(W_n)$ for $n\geq 11$.

Theorem 2.7 *For the Wheel* $W_n=K_1+C_{n-1}$,

$$\gamma_{cs}(W_n)=\begin{cases}4 & \textit{if } n \textit{ is even}\\ 3 & \textit{if } n \textit{ is odd}\end{cases}$$

Proof Let $v_1, v_2, \ldots, v_{n-1}, v$ be the vertices of W_n, where v is the center of the Wheel.

Case i n is even.

Define $f : V(W_n) \longrightarrow \{+1, -1\}$ by $f(v_1) = f(v_2) = f(v_3) = f(v_4) = f(v_5) = +1$ and for $6 \le i \le n-1$,

$$f(v_i) = \begin{cases} -1 & \text{if } i \text{ is even,} \\ +1 & \text{if } i \text{ is odd} \end{cases}$$

and $f(v) = -1$. We claim that f is a complementary signed dominating function.

$$\sum_{u \notin N[v_1]} f(u) = 5 - 2 + \left[\left(\frac{n-6}{2}\right) - 1\right] - \left(\frac{n-6}{2}\right) = 2.$$

For $i = 2,\ 3,\ 4$

$$\sum_{u \notin N[v_i]} f(u) = 5 - 3 + \left(\frac{n-6}{2}\right) - \left(\frac{n-6}{2}\right) = 2,$$

$$\sum_{u \notin N[v_5]} f(u) = 5 - 2 + \left(\frac{n-6}{2}\right) - \left[\left(\frac{n-6}{2}\right) - 1\right] = 4,$$

$$\sum_{u \notin N[v_6]} f(u) = 5 - 1 + \left(\frac{n-6}{2}\right) - 1 - \left[\left(\frac{n-6}{2}\right) - 1\right] = 4.$$

If i is odd and $7 \le i \le n-3$, then

$$\sum_{u \notin N[v_i]} f(u) = 5 + \left(\frac{n-6}{2}\right) - 1 - \left[\left(\frac{n-6}{2}\right) - 2\right] = 5 - 1 + 2 = 6.$$

If i is even and $8 \le i < n-1$, then

$$\sum_{u \notin N[v_i]} f(u) = 5 + \left(\frac{n-6}{2}\right) - 2 - \left[\left(\frac{n-6}{2}\right) - 1\right] = 4.$$

Also

$$\sum_{u \notin N[v_{n-1}]} f(u) = 5 - 1 + \left(\frac{n-6}{2}\right) - 1 - \left[\left(\frac{n-6}{2}\right) - 1\right] = 4.$$

Therefore f is a complementary signed dominating function. Since $\sum\limits_{u \notin N[v_4]} f(u) = 2$, the labeling is minimum with respect to the vertices $v_1,\ v_2,\ v_6, \ldots,\ v_{n-1}$. If $f(v_1) = -1$, then $\sum\limits_{u \notin N[v_3]} f(u) = 0$. It is easy to observe that

$$\sum_{u \in V(W_n)} f(u) = 5 + \left(\frac{n-6}{2}\right) - \left(\frac{n-6}{2}\right) - 1 = 4$$

is minimum. Hence $\gamma_{cs}(W_n) = 4$ if n is even.

Case ii n is odd.

Define $f : V(W_n) \longrightarrow \{+1, \ -1\}$ by $f(v_1) = f(v_2) = f(v_3) = f(v_4) = +1$ and for $5 \leq i \leq n-1$,

$$f(v_i) = \begin{cases} -1 & \text{if } i \text{ is even,} \\ +1 & \text{if } i \text{ is odd} \end{cases}$$

and $f(v) = -1$. We claim that f is a complementary signed dominating function.

$$\sum_{u \notin N[v_1]} f(u) = 4 - 2 + \left(\frac{n-5}{2}\right) - \left[\left(\frac{n-5}{2}\right) - 1\right] = 3.$$

For $i = 2, \ 3$

$$\sum_{u \notin N[v_i]} f(u) = 4 - 3 + \left(\frac{n-5}{2}\right) - \left(\frac{n-5}{2}\right) = 1,$$

$$\sum_{u \notin N[v_4]} f(u) = 4 - 2 + \left(\frac{n-5}{2}\right) - 1 - \left(\frac{n-5}{2}\right) = 1,$$

$$\sum_{u \notin N[v_5]} f(u) = 4 - 1 + \left(\frac{n-5}{2}\right) - 1 - \left[\left(\frac{n-5}{2}\right) - 1\right] = 3.$$

If i is even and $6 \leq i \leq n-3$, then

$$\sum_{u \notin N[v_i]} f(u) = 4 + \left(\frac{n-5}{2}\right) - 2 - \left[\left(\frac{n-5}{2}\right) - 1\right] = 3.$$

If i is odd and $5 < i < n-1$, then

$$\sum_{u \notin N[v_i]} f(u) = 4 + \left(\frac{n-5}{2}\right) - 1 - \left[\left(\frac{n-5}{2}\right) - 2\right] = 5,$$

$$\sum_{u \notin N[v_{n-1}]} f(u) = 4 - 1 + \left(\frac{n-5}{2}\right) - 1 - \left[\left(\frac{n-5}{2}\right) - 1\right] = 3.$$

Therefore f is a complementary signed dominating function. Since $\sum\limits_{u \notin N[v_2]} f(u) = 1$, the labeling is minimum with respect to the vertices $v_4, \ v_5, \ldots, \ v_{n-1}$. If $f(v_5) = -1$, then $\sum\limits_{u \notin N[v_3]} f(u) < 0$. It is easy to observe that $\sum\limits_{u \in V(W_n)} f(u) = 3$ is minimum. Hence $\gamma_{cs}(W_n) = 3$ if n is odd. □

Theorem 2.8 *For the wheel $W_n = K_1 + C_{n-1}$, $n \geq 4$, $\gamma_{cs}(W_n) = \gamma_{cs}(C_{n-1}) - 1$.*

Proof Let $v_1, v_2, \ldots, v_n, v$ be the vertices of W_n. Now,

$$\begin{aligned} \gamma_{cs}(W_n) &= \sum_{i=1}^{n-1} f(v_i) + f(v) \\ &= \gamma_{cs}(C_{n-1}) - 1 \end{aligned}$$

Hence $\gamma_{cs}(W_n) = \gamma_{cs}(C_{n-1}) - 1$. □

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Math.Combin.Book Ser. Vol.3 (2011), 47-54

On Dynamical Chaotic Weyl Representations of the Vacuüm C Metric and Their Retractions

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Abstract: In this article we will introduce the dynamical chaotic vacuüm C metric when $m = 0$. The relations between the dynamical chaotic vacuüm and its deformation retract ar obtained. Many types of dynamical chaotic vacuüm are deduced. The end limits of n-dimensional chaotic vacuüm are presented.

Key Words: Chaotic vacuüm, Retraction, deformation retract.

AMS(2010): 54C15

§1. Introduction

Chaos Theory is the qualitative study of unstable aperiodic behavior in deterministic nonlinear dynamical systems. Aperiodic behavior is observed when there is no variable, describing the state of the system, that undergoes a regular repetition of values. Unstable aperiodic behavior is highly complex it never repeats and it continues to manifest the effects of any small perturbation. As per the current mathematical theory a chaotic system is defined as showing sensitivity to initial conditions. In other words, to predict the future state of a system with certainty, you need to know the initial conditions with infinite accuracy, since errors increase rapidly with even the slightest inaccuracy. This is why the weather is so difficult to forecast. The theory also has been applied to business cycles, and dynamics of animal populations, as well as in fluid motion, planetary orbits, electrical currents in semi-conductors, medical conditions like epileptic seizures, and the modeling of arms races.

During the 1960s Edward Lorenz, a meteorologist at MIT, worked on a project to simulate weather patterns on a computer. He accidentally stumbled upon the butterfly effect after deviations in calculations off by thousandths greatly changed the simulations. The Butterfly Effect reflects how changes on the small-scale, can influence things on the large-scale. It is the classic example of chaos, where small changes may cause large changes. A butterfly, flapping its wings in Hong Kong, may change tornado patterns in Texas.

Chaos Theory regards organizations businesses as complex, dynamic, non-linear, co-creative and far-from-equilibrium systems. Their future performance cannot be predicted by past and

[1]Received November 25, 2010. Accepted August 18, 2011.

present events and actions. In a state of chaos, organizations behave in ways which are simultaneously both unpredictable chaotic and patterned orderly [6,10,11].

The vacuüm C-metric was first discovered by Levi-Civita within a class of degenerate static vacuüm metrics. However, over the years it has been rediscovered many times: by Newman and Tamburino, by Robinson and Trautman and again by Ehlers and Kundt who called it the C-metric in 1962. The charged C-metric has been studied in detail by Kinnersley and Walker. In general the space-time represented by the C-metric contains one or, via an extension, two uniformly accelerated particles as explained in. A description of the geometric properties of various extensions of the C-metric as well as a more complete list of references is contained in . The main property of the C-metric is the existence of two hypersurface-orthogonal Killing vectors, one of which is time like (showing the static property of the metric) in the space-time region of interest in this work. The C-metric is a vacuüm solution of the Einstein equations of the Petrov type D. Kinnersley and Walker showed that it represents black holes uniformly accelerated by nodal singularities in opposite directions along the axis of the axial symmetry [5,7,9].

Many types of dynamical manifolds And systems are discussed in [1-4,11]. A dynamical system in the space X is a function $q = f(p,t)$ which assigns to each point p of the space X and to each real number t, $\infty < t < \infty$ a definite point $q \in X$ and possesses the following three properties:

a – Initial condition: $f(p,0) = p$ for any point $p \in X$;

b – Property of continuity in both arguments simultaneously:

$$\lim_{\substack{p\to p_0\\ t\to t_0}} f(p,t) = f(p_0,t_0).$$

c – Group property $f(f(p,t_1),t_2) = f(p,t_1+t_2)$ [11].

A subset A of a topological space X is called a retract of X if there exists a continuous map $r : X \to A$ (called a retraction) such that $r(a) = a$, $\forall a \in A$ [8]. A subset A of a topological space X is a deformation retract of X if there exists a retraction $r : X \to A$ and a homotopy $f : X \times I \to X$ such that $f(x,0) = x, f(x,1) = r(x)$, $\forall x \in X$ and $f(a,t) = a$, $\forall a \in A, t \in [0,1]$ [8].

§2. Main Results

In this paper we will discuss some types of retractions and deformations retracts in Weyl representation of the space-time of the vacuüm C metric when $m = 0$.
The chaotic vacuüm C metric when $m = 0$ is defined as

$$ds^2 = \frac{1}{A^2(x(t)+y(t))^2}\begin{bmatrix} -k^2A^2(-1+y^2(t))du^2(t) + \frac{1}{1-x^2(t)}dx^2(t)+ \\ \frac{1}{-1+y^2(t)}dy^2(t) + \frac{1-x^2(t)}{k^2}dw^2(t) \end{bmatrix} \tag{1}$$

where $x(t), y(t), u(t), w(t)$ are functions of time. The chaotic Weyl coordinates system are

$$\begin{aligned} z(t) &= \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \\ r(t) &= \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2} \\ w(t) &= w(t) \\ u(t) &= u(t) \end{aligned}$$

Now we will use the following Lagrangian equations:

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \Psi_i'}\right) - \frac{\partial T}{\partial \Psi_i} = 0,\ i = 1, 2, 3, 4. \tag{2}$$

To deduce a chaotic geodesic which is a retraction of $\overset{ch}{C_0}$ by using Lagrangian equations, where $\overset{ch}{C_0}$ is the chaotic vacuüm C metric when $m = 0$. Since, $T = \frac{1}{2}ds^2$ it follows that

$$T = \frac{1}{2}\left(\frac{1}{A^2(x(t)+y(t))^2}\begin{bmatrix} -k^2A^2(-1+y^2(t))du^2(t) + \frac{1}{1-x^2(t)}dx^2(t) + \\ \frac{1}{-1+y^2(t)}dy^2(t) + \frac{1-x^2(t)}{k^2}dw^2(t) \end{bmatrix}\right). \tag{3}$$

Then the Lagrangian equations of chaotic vacuüm $\overset{ch}{C_0}$ are

$$\frac{d}{ds}\left[-\frac{k^2(-1+y^2(t)}{(x(t)+y(t))^2})u'(t)\right] = 0 \tag{4}$$

$$\frac{d}{ds}\left[\frac{1-x^2(t)}{k^2A^2(x(t)+y(t))^2}w'(t)\right] = 0 \tag{5}$$

$$\begin{aligned} &\frac{d}{ds}\left[\frac{x'(t)}{A^2(x(t)+y(t))^2(1-x^2(t))}\right] - \frac{1}{A^2(x(t)+y(t))^2}\times \\ &\qquad \left[\frac{x(t)}{1-x^2(t)}(x'(t))^2 + \frac{-x(t)}{k^2}(w'(t))^2\right] + \\ &\begin{bmatrix} -k^2A^2(-1+y^2(t))(u'(t))^2 + \frac{1}{1-x^2(t)}(x'(t))^2 + \\ \frac{1}{-1+y^2(t)}(y'(t))^2 + \frac{1-x^2(t)}{k^2}(w'(t))^2 \end{bmatrix}\left[\frac{1}{A^2(x(t)+y(t))^3}\right] = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{d}{ds}\left[\frac{y'(t)}{A^2(x(t)+y(t))^2(-1+y^2(t))}\right] - \frac{1}{A^2(x(t)+y(t))^2}\times \\ &\qquad \left[-k^2A^2y(t)(u'(t))^2 - \frac{y(t)}{(-1+y^2(t))^2}(y'(t))^2\right] + \\ &\begin{bmatrix} -k^2A^2(-1+y^2(t))(u'(t))^2 + \frac{1}{1-x^2(t)}(x'(t))^2 + \\ \frac{1}{-1+y^2(t)}(y'(t))^2 + \frac{1-x^2(t)}{k^2}(w'(t))^2 \end{bmatrix}\left[\frac{1}{A^2(x(t)+y(t))^3}\right] = 0 \end{aligned} \tag{7}$$

From Eq.(2.4), we obtain $\frac{k^2(-1+y^2(t)}{(x(t)+y(t))^2})u'(t)$ =constant ,say λ,if $\lambda=0$ then $u'(t)=0$, and so $u(t)$ =constant α, if $\alpha=0$ we have the following retraction

$$\begin{aligned} z(t) &= \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \\ r(t) &= \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2} \\ w(t) &= w(t) \\ u(t) &= 0 \end{aligned}$$

which is the chaotic retraction $\overset{ch}{C}_{01}$ in the chaotic vacuüm $\overset{ch}{C}_{0}$. Also, from Eq.(2.5), we get $\frac{1-x^2(t)}{k^2A^2(x(t)+y(t))^2}w'(t)$ =constant ,say ν , if $\nu=0$ then $w'(t)=0$, and so $w(t)$ =constant δ, if $\delta=0$ we have the following retraction

$$\begin{aligned} z(t) &= \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \\ r(t) &= \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2} \\ w(t) &= 0 \\ u(t) &= u(t) \end{aligned}$$

which is the chaotic retraction $\overset{ch}{C}_{02}$ in the chaotic vacuüm $\overset{ch}{C}_{0}$. Moreover from Eq.(2.6), we have $\frac{d}{ds}\left[\frac{x'(t)}{A^2(x(t)+y(t))^2(1-x^2(t))}\right]$ =constant , say ϖ , if $\varpi=0$ then $x'(t)=0$, and so $x(t)$ =constant β, if $\beta=0$ we have the following retraction

$$\begin{aligned} z(t) &= \frac{1}{Ay^2(t)} \\ r(t) &= \frac{(y^2(t)-1)^{\frac{1}{2}}}{Ay^2(t)} \\ w(t) &= w(t) \\ u(t) &= u(t) \end{aligned}$$

which is chaotic geodesic $\overset{ch}{C}_{03}$ in chaotic hyper affine subspace of chaotic vacuüm $\overset{ch}{C}_{0}$. Now, from Eq.(2.7), we have $\frac{d}{ds}\left[\frac{y'(t)}{A^2(x(t)+y(t))^2(-1+y^2(t))}\right]$ =constant ,say γ, if $\gamma=0$ then $y'(t)=0$, and so $y(t)$ =constant ρ, if $\rho=0$ we have the following retraction

$$\begin{aligned} z(t) &= \frac{1}{Ax^2(t)} \\ r(t) &= \frac{i(1-x^2(t))^{\frac{1}{2}}}{Ax^2(t)} \\ w(t) &= w(t) \\ u(t) &= u(t) \end{aligned}$$

which is chaotic geodesic $\overset{ch}{C}_{04}$ in chaotic hyper affine subspace of chaotic vacuüm $\overset{ch}{C}_0$

From the above discussion we can formulate the following theorem.

Theorem 2.1 *The geodesic of the chaotic vacuüm $\overset{ch}{C}_0$ by using Lagrangian equations is a type of retraction which is chaotic hyper affine subspace of $\overset{ch}{C}_0$.*

Now we will discuss the relations between the deformation retracts of chaotic vacuüm and their geodesics. The deformation retract of the chaotic vacuüm $\overset{ch}{C}_0$ is defined as $\Psi : \overset{ch}{C}_0 \times I \to \overset{ch}{C}_0$,where I is the closed interval $[0, 1]$. The retraction of the chaotic vacuüm $\overset{ch}{C}_0$ is defined as $r : \overset{ch}{C}_0 \to \overset{ch}{C}_{01}, \overset{ch}{C}_{02}, \overset{ch}{C}_{03}$ and $\overset{ch}{C}_{04}$. The deformation retract of the chaotic vacuüm $\overset{ch}{C}_0$ into a geodesic $\overset{ch}{C}_{01} \subseteq \overset{ch}{C}_0$ is given by

$$\begin{aligned}\Psi(m,s) &= \cos(\frac{\pi s}{2})\left(\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t)\right)\\ &\quad + \sin(\frac{\pi s}{2})\left(\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), 0\right)\end{aligned}$$

and so $\Psi(m, 0) = \overset{ch}{C}_0$, $\Psi(m, 1) = \overset{ch}{C}_{01}$. The deformation retract of the chaotic vacuüm $\overset{ch}{C}_0$ into a geodesic $\overset{ch}{C}_{02} \subseteq \overset{ch}{C}_0$ is given by

$$\begin{aligned}\Psi(m,s) &= \cos(\frac{\pi s}{2})\left(\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t)\right)\\ &\quad + \sin(\frac{\pi s}{2})\left(\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, 0, u(t)\right).\end{aligned}$$

Thus $\Psi(m, 0) = \overset{ch}{C}_0$, $\Psi(m, 1) = \overset{ch}{C}_{02}$. The deformation retract of the chaotic vacuüm $\overset{ch}{C}_0$ into a geodesic $\overset{ch}{C}_{03} \subseteq \overset{ch}{C}_0$ is given by

$$\begin{aligned}\Psi(m,s) &= \cos(\frac{\pi s}{2})\left(\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t)\right)\\ &\quad + \sin(\frac{\pi s}{2})\left(\frac{1}{Ay^2(t)}, \frac{(y^2(t)-1)^{\frac{1}{2}}}{Ay^2(t)}, w(t), u(t)\right).\end{aligned}$$

So $\Psi(m, 0) = \overset{ch}{C}_0$, $\Psi(m, 1) = \overset{ch}{C}_{03}$. The deformation retract of the chaotic vacuüm $\overset{ch}{C}_0$ into a geodesic $\overset{ch}{C}_{04} \subseteq \overset{ch}{C}_0$ isgiven by

$$\begin{aligned}\Psi(m,s) &= \cos(\frac{\pi s}{2})\left(\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t)\right)\\ &\quad + \sin(\frac{\pi s}{2})\left(\frac{1}{Ax^2(t)}, \frac{i(1-x^2(t))^{\frac{1}{2}}}{Ax^2(t)}, w(t), u(t)\right),\end{aligned}$$

and so $\Psi(m,0)=\overset{ch}{C_0}$, $\Psi(m,1)=\overset{ch}{C}_{04}$.

Theorem 2.2 *The end limit of dynamical chaotic n-dimensional vacuüm $\overset{ch}{V_n}$ is zero-dimensional chaotic vacuüm $\overset{ch}{V_0}$.*

Proof Let D_i be the dynamical chaotic n-dimensional vacuüm $\overset{ch}{V_n}$ .Then we get the following chains:

$$\overset{ch}{V_n}\overset{D_1^1}{\to}\overset{ch}{V_n^1}\overset{D_2^1}{\to}\overset{ch}{V_n^2}\to\cdots\to\overset{ch}{V_n^{m-1}} \text{ such that } \lim_{m\to\infty}D_m^1(\overset{ch}{V_n^{m-1}})=\overset{ch}{V}_{n-1};$$
$$\overset{ch}{V}_{n-1}\overset{D_1^1}{\to}\overset{ch}{V}_{n-1}^1\overset{D_2^1}{\to}\overset{ch}{V}_{n-1}^2\to\cdots\to\overset{ch}{V}_{n-1}^{m-1} \text{ such that } \lim_{m\to\infty}D_m^1(\overset{ch}{V}_{n-1}^{m-1})=\overset{ch}{V}_{n-2},$$
$$\vdots$$
$$\overset{ch}{V_1}\overset{D_1^1}{\to}\overset{ch}{V_1^1}\overset{D_2^1}{\to}\overset{ch}{V_1^2}\to\cdots\to\overset{ch}{V_1^{m-1}} \text{ such that } \lim_{m\to\infty}D_m^1(\overset{ch}{V_1^{m-1}})=\overset{ch}{V_0}.$$

Therefore, from the last chain the end limits of the dynamical chaotic n-dimensional vacuüm $\overset{ch}{V_n}$ is zero-dimensional chaotic vacuüm. □

Now we are going to discuss some types of dynamical chaotic vacuüm $\overset{ch}{C_0}$. Let $D:\overset{ch}{C_0}\to\overset{ch}{C_0}$ be the dynamical chaotic vacuüm on $\overset{ch}{C_0}$ which preserve the isometry of chaotic vacuüm $\overset{ch}{C_0}$ into itself such that $D(x_1,x_2,x_3,x_4)=(|x_1|,x_2,x_3,x_4)$. So we can define D as

$$D:\left(\frac{1+x(t)y(t)}{A(x(t)+y(t))^2},\frac{(1-x^2(t))^{\frac12}(y^2(t)-1)^{\frac12}}{A(x(t)+y(t))^2},w(t),u(t)\right)$$
$$\longrightarrow\left(\left|\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}\right|,\frac{(1-x^2(t))^{\frac12}(y^2(t)-1)^{\frac12}}{A(x(t)+y(t))^2},w(t),u(t)\right).$$

The deformation retracts of the dynamical chaotic vacuüm $\overset{ch}{C_0}$ into the dynamical chaotic retraction $\overset{ch}{C}_{01}\subseteq\overset{ch}{C_0}$ is given by

$$\Psi_D:\left(\left|\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}\right|,\frac{(1-x^2(t))^{\frac12}(y^2(t)-1)^{\frac12}}{A(x(t)+y(t))^2},w(t),u(t)\right)\times I$$
$$\longrightarrow\left(\left|\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}\right|,\frac{(1-x^2(t))^{\frac12}(y^2(t)-1)^{\frac12}}{A(x(t)+y(t))^2},w(t),0\right)$$

with

$$\Psi_D\ (m,s) = \cos(\frac{\pi s}{2})\left(\left|\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}\right|,\frac{(1-x^2(t))^{\frac12}(y^2(t)-1)^{\frac12}}{A(x(t)+y(t))^2},w(t),u(t)\right)$$
$$+\sin(\frac{\pi s}{2})\left(\left|\frac{1+x(t)y(t)}{A(x(t)+y(t))^2}\right|,\frac{(1-x^2(t))^{\frac12}(y^2(t)-1)^{\frac12}}{A(x(t)+y(t))^2},w(t),0\right).$$

The deformation retracts of the dynamical chaotic vacuüm $\overset{ch}{C_0}$ into the dynamical chaotic

retraction $\overset{ch}{C}_{02} \subseteq \overset{ch}{C}_0$ is given by

$$\Psi_D : \left(\left| \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \right|, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t) \right) \times I$$

$$\longrightarrow \left(\left| \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \right|, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, 0, u(t) \right)$$

with

$$\begin{aligned} \Psi_D\ (m,s) &= \cos(\frac{\pi s}{2}) \left(\left| \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \right|, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t) \right) \\ &\quad + \sin(\frac{\pi s}{2}) \left(\left| \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \right|, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, 0, u(t) \right). \end{aligned}$$

The deformation retracts of the dynamical chaotic vacuüm $\overset{ch}{C}_0$ into the dynamical chaotic geodesic $\overset{ch}{C}_{03} \subseteq \overset{ch}{C}_0$ is given by

$$\Psi_D : \left(\left| \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \right|, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t) \right) \times I$$

$$\longrightarrow \left(\left| \frac{1}{Ay^2(t)} \right|, \frac{(y^2(t)-1)^{\frac{1}{2}}}{Ay^2(t)}, w(t), u(t) \right)$$

with

$$\begin{aligned} \Psi_D\ (m,s) &= \cos(\frac{\pi s}{2}) \left(\left| \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \right|, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t) \right) \\ &\quad + \sin(\frac{\pi s}{2}) \left(\left| \frac{1}{Ay^2(t)} \right|, \frac{(y^2(t)-1)^{\frac{1}{2}}}{Ay^2(t)}, w(t), u(t) \right). \end{aligned}$$

The deformation retracts of the dynamical chaotic vacuüm $\overset{ch}{C}_0$ into the dynamical chaotic geodesic $\overset{ch}{C}_{04} \subseteq \overset{ch}{C}_0$ is given by

$$\Psi_D : \left(\left| \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \right|, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t) \right) \times I$$

$$\longrightarrow \left(\left| \frac{1}{Ax^2(t)} \right|, \frac{i(1-x^2(t))^{\frac{1}{2}}}{Ax^2(t)}, w(t), u(t) \right)$$

with

$$\begin{aligned} \Psi_D\ (m,s) &= \cos(\frac{\pi s}{2}) \left(\left| \frac{1+x(t)y(t)}{A(x(t)+y(t))^2} \right|, \frac{(1-x^2(t))^{\frac{1}{2}}(y^2(t)-1)^{\frac{1}{2}}}{A(x(t)+y(t))^2}, w(t), u(t) \right) \\ &\quad + \sin(\frac{\pi s}{2}) \left(\left| \frac{1}{Ax^2(t)} \right|, \frac{i(1-x^2(t))^{\frac{1}{2}}}{Ax^2(t)}, w(t), u(t) \right). \end{aligned}$$

Then the following theorem has been proved.

Theorem 2.3 *The deformation retracts of the dynamical chaotic vacuüm* $\overset{ch}{C_0}$ *into chaotic geodesic is different from the deformation retracts of the chaotic vacuüm* $\overset{ch}{C_0}$ *into the chaotic geodesic.*

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Math.Combin.Book Ser. Vol.3 (2011), 55-60

Bounds for Distance$-g$ Domination Parameters in Circulant Graphs

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Abstract: A circulant graph is a Cayley graph constructed out of a finite cyclic group Γ and a generating set A is a subset of Γ. In this paper, we attempt to find upper bounds for distance-g domination , distance-g paired domination and distance-g connected domination number for circulant graphs. Exact values are also determined in certain cases.

Key Words: Circulant graph, Smarandachely distance-g paired-(U, V) dominating $\mathscr{P}$-set, distance-g domination, distance-g paired, total and connected domination, distance-g efficient domination.

AMS(2010): 05C69

§1. Introduction

Let Γ be a finite group with e as the identity. A *generating set* of the group Γ is a subset A such that every element of Γ can be expressed as the product of finitely many elements of A. Assume that $e \notin A$ and $a \in A$ implies $a^{-1} \in A$. The *Cayley graph* $G = (V, E)$, where $V(G) = \Gamma$ and $E(G) = \{(x, xa) | x \in V(G), a \in A\}$ and it is denoted by $Cay(\Gamma, A)$. The exclusion of e from A eliminates the possibility of loops in the graph. When $\Gamma = Z_n$, the Cayley graph $Cay(\Gamma, A)$ is called as *circulant graph* and denoted by $Cir(n, A)$.

Suppose $G = (V, E)$ is a graph, the open neighbourhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of vertices adjacent to v. The closed neighbourhood of v is $N[v] = N(v) \cup \{v\}$. For a set $D \subseteq V$, the open neighbourhood $N(D)$ is defined to be $\bigcup\limits_{v \in D} N(v)$, and the closed neighbourhood of D is $N[D] = N(D) \cup D$. Let $u, v \in V(G)$, then $d(u, v)$ is the length of the shortest $uv-$path. For any $v \in V(G), N^g(v) = \{u \in V(G) : d(u, v) \leq g\}$ and $N^g[v] = N^g(v) \cup \{v\}$. A set $D \subseteq V$,

[1]Received February 12, 2011. Accepted August 20, 2011.

of vertices in G is called a *dominating set* if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. That is $N[D] = V(G)$. The domination number $\gamma(G)$ of G is the minimum cardinality among all the dominating sets in G and the corresponding dominating set is called a γ-set. A set $D \subseteq V$, of vertices in G is called a *distance-g dominating set* if $N^g[D] = V(G)$. The distance$-g$ domination number $\gamma^g(G)$ of G is the minimum cardinality among all the distance$-g$ dominating sets in G and the corresponding distance$-g$ dominating set is called a γ^g-set.

Let G be a graph, $D,\ U,\ V \subset V(G)$ with $U \bigcup V = V(G)$, $U \bigcap V = \emptyset$, $g \geq 1$ an integer and $\langle D\rangle_G$ having graphical property $\mathscr{P}$. If $d(u, D) \leq g$ for $u \in U - D$ but $d(v, D) > g$ for $v \in V - D$, such a vertex subset D is called a *Smarandachely distance-g paired-(U, V) dominating $\mathscr{P}$-set*. Particularly, if $U = V(G)$, $V = \emptyset$ and $\mathscr{P}$=perfect matching, i.e., a Smarandachely distance-g paired-$(V(G), \emptyset)$ dominating $\mathscr{P}$-set D is called a *distance-g paired dominating set*. The minimum cardinality among all the distance-g paired dominating sets for graph G is the distance-g paired domination number, denoted by $\gamma_p^g(G)$. A set $S \subseteq V$, of vertices in G is called a *distance-g total dominating set* if $N^g(S) = V(G)$. The distance$-g$ total domination number $\gamma_t^g(G)$ of G is the minimum cardinality among all the distance $-g$ total dominating sets in G and the corresponding distance$-g$ total dominating set is called a γ_t^g-set. A set $D \subseteq V$, of vertices in G is said to be *distance-g connected dominating set* if every vertex in $V(G) - D$ is within distance g of a vertex in D and the induced subgraph $< D >$ is $g-$ connected (If $x \in N_g[y]$ for all $x, y \in D$, then x and y are $g-$connected). The minimum cardinality of a distance $-g$ connected dominating set for a graph G is the distance $-g$ connected domination number, denoted by $\gamma_c^g(G)$. A set $D \subseteq V$ is called a *distance-g efficient dominating set* if for every vertex $v \in V$, $|N^g[v] \cap D| = 1$.

The concept of domination for circulant graphs has been studied by various authors and one can refer to [1,6-8] and Rani [9-11] obtained the various domination numbers including total, connected and independent domination numbers for Cayley graphs on Z_n. Paired domination was introduced by Haynes and Slater. In 2008, Joanna Raczek [2] generalized the paired domination and investigated properties of the distance paired domination number of a path, cycle and some non-trivial trees. Raczek also proved that distance$-g$ paired domination problem is NP-complete. Haoli Wang et al. [3] obtained distance$-g$ paired domination number of circulant graphs for a particular kind of generating set. In this paper, we attempt to find the sharp upper bounds for distance$-g$ paired domination number for circulant graphs for a general generating set. The distance version of domination have a strong background of applications. For instance, efficient construction of distance$-g$ dominating sets can be applied in the context of distributed data structure, where it is proposed that distance$-g$ dominating sets can be selected for locating copies of a distributed directory. Also it is useful for efficient selection of network centers for server placement.

Throughout this paper, n is a fixed positive integer, $\Gamma = \mathbb{Z}_n$, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n - a_k, n - a_{k-1}, \ldots, n - a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$, $A_1 = \{a_1, a_2, \ldots, a_k\}$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$ and $d = \max\limits_{1 \leq i \leq k} \{d_i\}$.

§2. Distance-g Domination

In this section, we obtain upper bounds for the distance$-g$ domination number and distance$-g$ efficient domination number. Also whenever the equality occurs we give the corresponding sets.

Theorem 2.1 *Let $n(\geq 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n-a_k, n-a_{k-1}, \ldots, n-a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$, and $G = Cir(n, A)$. If $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d = \max_{1\leq i\leq k}\{d_i\}$, then $\gamma^g(G) \leq d\lceil \frac{n}{2ga_k + d} \rceil$.*

Proof Let $x = 2ga_k + d$ and $\ell = \lceil \frac{n}{x} \rceil$. Consider the set $D = \{0, 1, \ldots, d-1, x, x+1, \ldots, x+d-1, 2x, 2x+1, \ldots, 2x+d-1, \ldots, (\ell-1)x, (\ell-1)x+1, \ldots, (\ell-1)x+d-1\}$. Note that $|D| = d\ell$ and $ra_i \in N^g[a_i]$, for $1 \leq r \leq g$. Let $v \in V(G)$. By division algorithm, one can write $v = ix + j$ for some i with $0 \leq i \leq \ell - 1$ and $0 \leq j \leq x - 1$. We have the following cases:

Case i Suppose $0 \leq i \leq \ell - 1$ and $0 \leq j \leq ga_k + d - 1$.

SubCase i When $0 \leq j < a_1$, then by the definition of d, $v \in D \subseteq N^g[D]$.

SubCase ii When $a_1 \leq j \leq ga_k + d - 1$, one can write $j = ra_m + t$, for some integers r, m, t with $1 \leq r \leq g$, $1 \leq m \leq k$ and $0 \leq t \leq d-1$ and so $v = ix + t + ra_m$ where as $ix + t \in D$. Since $ra_m \in N^g[a_m]$, we get $v \in N^g[\{ix, ix+1, \ldots, ix+(d-1)\}] \subseteq N^g[D]$.

Case ii Suppose $0 \leq i \leq \ell - 2$ and $ga_k + d \leq j \leq 2ga_k + d - 1$. Choose an integer h with $1 \leq h \leq ga_k$ such that $v + h = (i+1)x$. One can write $h = ra_m - t$, for some integers r, m, t with $1 \leq r \leq g$, $1 \leq m \leq k$ and $0 \leq t \leq d - 1$ and hence $v + ra_m = (i+1)x + t$, which means that $v \in N^g[\{(i+1)x, (i+1)x+1, \ldots, (i+1)x+(d-1)\}] \subseteq N^g[D]$.

Case iii Suppose $i = \ell - 1$ and $ga_k + d \leq j \leq 2ga_k + d - 1$. As mentioned earlier, one can choose an integer h with $1 \leq h \leq ga_k$ such that $v + h = 0$. Write $h = ra_m - t$ with $1 \leq r \leq m$, $1 \leq m \leq k$ and $0 \leq t \leq d-1$, which means that $v \in N^g[\{0, 1, 2, \ldots, d-1\}] \subseteq N^g[D]$. Thus D is a distance$-g$ dominating set of G. □

Theorem 2.2 *Let $n(\geq 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{d, 2d, \ldots, kd, n-kd, n-(k-1)d, \ldots, n-d\}$ and $G = Cir(n, A)$. If $d(1+2gk)$ divides n, then $\gamma^g(G) = \frac{n}{1+2gk}$. In this case, $Cir(n, A)$ has a distance-g efficient dominating set.*

Proof In the notation of the Theorem 2.1, $a_i = id$ for all $1 \leq i \leq k$ and so $d_i = d$. By Theorem 2.1, $D = \{0, 1, \ldots, d-1, x, x+1, \ldots, x+(d-1), 2x, 2x+1, \ldots, 2x+(d-1), \ldots (\ell-1)x, (\ell-1)x+1, \ldots, (\ell-1)x+(d-1)\}$ is a distance-g dominating set and hence $\gamma^g(G) \leq d(\frac{n}{d(1+2gk)}) = \frac{n}{1+2gk}$. Let $n = \ell(d(1+2gk))$. Since $|N^g[v]| = 2gk+1$, for all $v \in V(G)$, $|D| = \ell d$ and $|N^g[u] \cap N^g[v]| = \emptyset$ for any two distinct vertices $u, v \in D$, we have $\gamma^g(G) = \frac{n}{1+2gk}$. From this, one can conclude that D is a distance-g efficient dominating set in G.

§3. Distance-g Paired Domination, Distance-g Connected Domination and Distance-g Total Domination

In this section, we obtain upper bounds for the distance$-g$ paired domination number, distance$-g$ connected domination number and distance$-g$ total domination number. Also whenever the equality occurs we give the corresponding sets.

Theorem 3.1 *Let $n(\geq 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n-a_k, n-a_{k-1}, \ldots, n-a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$, and $G = Cir(n, A)$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d = \max\limits_{1\leq i\leq k}\{d_i\}$. If $(2g+1)a_k + d$ divides n, then $\gamma_p^g(G) \leq 2d(\frac{n}{(2g+1)a_k + d})$.*

Proof Let $x = (2g+1)a_k + d$, $\ell = \frac{n}{x}$ and $D_p = \{0, 1, \ldots, d-1, a_k, a_k+1, \ldots, a_k+(d-1), x, x+1, \ldots, x+(d-1), a_k+x, a_k+x+1, \ldots, a_k+x+(d-1), \ldots, (\ell-1)x, (\ell-1)x+1, \ldots, (\ell-1)x+(d-1), a_k+(\ell-1)x, a_k+(\ell-1)x+1, \ldots, a_k+(\ell-1)x+(d-1)\}$. Note that $|D_p| = 2d\ell$ and $ra_i \in N^g[a_i]$ for $1 \leq r \leq g$. Let $v \in V(G)$. By division algorithm, one can write $v = ix + j$ for some i, j with $0 \leq i \leq \ell-1$ and $0 \leq j \leq x-1$. We have the following cases:

Case i Suppose $0 \leq i \leq \ell-1$ and $0 \leq j \leq ga_k + (d-1)$.

SubCase i If $0 \leq j < a_1$ then by the definition of d, $v \in N^g[D_p]$.

SubCase ii When $a_1 \leq j \leq ga_k + d - 1$, one can write $j = ra_m + t$, for $1 \leq r \leq g$, $1 \leq m \leq k$ and $0 \leq t \leq d-1$, then $v = ix + ra_m + t$ and so $v \in N^g[\{ix, ix+1, \ldots, ix+(d-1)\}] \subseteq N^g[D_p]$.

Case ii Suppose $0 \leq i \leq \ell-1$ and $ga_k + d \leq j \leq ga_k + a_k + d - 1$. In this case v can be written as $v = ix + ga_k + h$ where $d \leq h \leq a_k + (d-1)$. By the property of vertex transitivity and by case(i), we have $v \in N^g[\{ix+a_k, ix+a_k+1, \ldots, ix+a_k+(d-1)\}] \subseteq N^g[D_p]$.

Case iii Suppose $0 \leq i \leq \ell-1$ and $ga_k + a_k + d \leq j \leq 2ga_k + a_k + d - 1$.

SubCase i Suppose $0 \leq i \leq \ell-2$. In this case v can be written as $v = (i+1)x + (j-x)$ for some i, j such that $0 \leq i \leq \ell-2$ and $-ga_k \leq j-x \leq 0$. Thus $v + (x-j) = (i+1)x$ and $0 \leq x-j \leq ga_k$. Hence by case (i), we have $v \in N^g[\{(i+1)x, (i+1)x+1, \ldots, (i+1)x+(d-1)\}] \subseteq N^g[D_p]$.

SubCase ii Suppose $i = \ell-1$. Then $v \in N^g[\{0, 1, \ldots, d-1\}] \subseteq N^g[D_p]$. Thus D_p is a distance-g dominating set of G. let $D' = \{0, 1, \ldots, d-1, x, x+1, \ldots, x+(d-1), \ldots, (\ell-1)x, (\ell-1)x+1, \ldots, (\ell-1)x+(d-1)\}$. It is note that $D' \subseteq D_p$ and for all $u \in D'$, there exists $v = u + a_k \in D_p$ such that u and v are adjacent in $< D_p >$. Hence $< D_p >$ has a perfect matching and D_p is a distance-g paired dominating set. □

Lemma 3.2 *let $n(\geq 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n-a_k, n-a_{k-1}, \ldots, n-a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$ and $G = Cir(n, A)$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d = \max\limits_{1\leq i\leq k}\{d_i\}$. Then $\gamma_t^g(G) \leq 2d\lceil \frac{n}{(2g+1)a_k + d} \rceil$.*

Proof Let $\ell = \lceil \frac{n}{(2g+1)a_k + d} \rceil$ and let $x = d + (2g+1)a_k$. Then $n = (\ell - 1)x + j$ for some $0 \leq j \leq x - 1$. As in the proof of Theorem 2.1, one can prove that $D_t = \{0, 1, \ldots, d-1, a_k, a_k + 1, \ldots, a_k + (d-1), x, x+1, \ldots, x + (d-1), a_k + x, a_k + x + 1, \ldots, a_k + x + (d-1), \ldots, (\ell - 1)x, (\ell-1)x+1, \ldots, (\ell-1)x+(d-1), a_k + (\ell-1)x, a_k + (\ell-1)x + 1, \ldots, a_k + (\ell-1)x + (d-1)\}$, is a distance-$g$ dominating set. Also note that, for every $z \in D_t$ there exists another adjacent vertex $z + a_k$ or $z - a_k \in D_t$. Thus D_t is a distance-g total dominating set. □

Now we obtain some equality for the distance g-paired domination number in certain classes of circulant graphs.

Corollary 3.3 *Let $n(\geq 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{1, 2, \ldots, k, n-k, \ldots, n-1\} \subset \mathbb{Z}_n$ and $G = Cir(n, A)$. Then $\gamma_p^g(G) = 2(\frac{n}{(2g+1)k+1})$.*

Proof Take $a_k = k$ in the statement of Theorem 3.1. As $d = 1$ and by Theorem 3.1, one can easily prove $D = \{0, k, x, x+k, \ldots, (\ell-1)x, (\ell-1)x+k\}$ is a distance-g paired dominating set and hence $\gamma_p^g(G) \leq 2(\frac{n}{(2g+1)k+1})$. Also,since any two adjacent vertices in D can dominate at most $(2g+1)k+1$ distinct vertices of $G, \gamma_p^g(G) \geq 2(\frac{n}{(2g+1)k+1})$. □

Remark 3.4 Joanna Raczek [2] has proved $\gamma_p^g(C_n) = 2\lceil \frac{n}{2g+2} \rceil$, for $n \geq 3$. This can be obtained by taking $a_k = 1$ and $d = 1$ in Theorem 3.1. Also, Haoli Wang et al. [3] have obtained the distance-g paired domination number for $Cir(n, A = \{1, k\})$ for $k = 2, 3$ and 4.

Remark 3.5 The upper bound obtained for distance-g paired domination number matches with the distance-g total domination number. i.e., $\gamma_t^g(G) \leq 2d\lceil \frac{n}{(2g+1)a_k + d} \rceil$. In general, for $Cir(n, A)$, the distance-g paired domination number is not equal to distance-g total domination, for all g.

Lemma 3.6 *Let $n(\geq 3)$ be a positive integer, $m = \lfloor \frac{n}{2} \rfloor$, k is an integer such that $1 \leq k \leq m$ and g is a fixed positive integer such that $1 \leq g \leq m$. Let $A = \{a_1, a_2, \ldots, a_k, n-a_k, n-a_{k-1}, \ldots, n-a_1\} \subset \mathbb{Z}_n$ with $1 \leq a_1 < a_2 < \ldots < a_k \leq m$, and $G = Cir(n, A)$. Let $d_1 = a_1, d_i = a_i - a_{i-1}$ for $2 \leq i \leq k$, $d = \max_{1 \leq i \leq k} \{d_i\}$, then $\gamma_c^g(G) \leq d(1 + \lceil \frac{n - (d + 2ga_k)}{(d-1) + ga_k} \rceil)$.*

Proof Let $\ell = \lceil \frac{n - (d + 2ga_k)}{(d-1) + ga_k} \rceil$ and $D_c = \{0, 1, \ldots, d-1, d-1+ga_k, d-1+ga_k+1, \ldots, d-1+ga_k+d-1, d-1+2ga_k, d-1+2ga_k+1, \ldots, d-1+2ga_k+d-1, \ldots, 2(d-1+ga_k), 2(d-1+ga_k)+1, \ldots, \ell(d-1+ga_k)+d-1, \ell(d-1+ga_k), \ell(d-1+ga_k)+1, \ldots, \ell(d-1+ga_k)+d-1\}$. As in the proof of Theorem 2.1, we can prove D_c is a distance $-g$ dominating set. Since $1 \in A$ and $ra_i \in N^g[a_i]$ for $1 \leq r \leq g$, $0 + j, d - 1 + ga_k + j, 2(d - 1 + ga_k) + j, \ldots, \ell(d - 1 + ga_k) + j$ are $-g$ connected in the induced subgraph $< D_c >$ for each j with $0 \leq j \leq d - 1$. Thus D_c is a distance $-g$ connected dominating set for G with $|D_c| = d(1 + \lceil \frac{n - (d + 2ga_k)}{(d-1) + ga_k} \rceil)$. □

Remark 3.7 From the above lemma, by replacing $g = 1$, we get the usual connected domination

number. i.e., when $g = 1$, $\gamma_c(G) \leq d(1 + \lceil \frac{n-(d+2a_k)}{(d-1)+a_k} \rceil)$.

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Math.Combin.Book Ser. Vol.3 (2011), 61-67

Surface Embeddability of Graphs via Reductions

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Abstract: On the basis of reductions, polyhedral forms of Jordan axiom on closed curve in the plane are extended to establish characterizations for the surface embeddability of a graph.

Key Words: Surface, graph, Smarandache λ^S-drawing, embedding, Jordan closed cure axiom, forbidden minor.

AMS(2010): 05C15, 05C25

§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache* λ^S*-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S.

The classical version of Jordan curve theorem in topology states that a single closed curve C separates the sphere into two connected components of which C is their common boundary. In this section, we investigate the polyhedral statements and proofs of the Jordan curve theorem.

Let $\Sigma = \Sigma(G; F)$ be a polyhedron whose underlying graph $G = (V, E)$ with F as the set of faces. If any circuit C of G not a face boundary of Σ has the property that there exist two proper subgraphs In and Ou of G such that

$$In \bigcup Ou = G; \ In \bigcap Ou = C, \qquad (A)$$

then Σ is said to have the *first Jordan curve property*, or simply write as 1-JCP. For a graph G, if there is a polyhedron $\Sigma = \Sigma(G; F)$ which has the 1-JCP, then G is said to have the 1-JCP as well.

Of course, in order to make sense for the problems discussed in this section, we always suppose that all the members of F in the polyhedron $\Sigma = \Sigma(G; F)$ are circuits of G.

Theorem A(First Jordan curve theorem) *G has the 1-JCP If, and only if, G is planar.*

Proof Because of $\mathcal{H}_1(\Sigma) = 0, \Sigma = \Sigma(G; F)$, from Theorem 4.2.5 in [1], we know that

[1]Received December 25, 2010. Accepted August 25, 2011.

Im $\partial_2 =$ Ker $\partial_1 = \mathcal{C}$, the cycle space of G and hence Im $\partial_2 \supseteq F$ which contains a basis of $\mathcal{C}$. Thus, for any circuit $C \notin F$, there exists a subset D of F such that

$$C = \sum_{f\in D} \partial_2 f;\ C = \sum_{f\in F\setminus D} \partial_2 f. \tag{B}$$

Moreover, if we write

$$Ou = G[\bigcup_{f\in D} f];\ In = G[\bigcup_{f\in F\setminus D} f],$$

then Ou and In satisfy the relations in (A) since any edge of G appears exactly twice in the members of F. This is the sufficiency.

Conversely, if G is not planar, then G only have embedding on surfaces of genus not 0. Because of the existence of non contractible circuit, such a circuit does not satisfy the 1-JCP and hence G is without 1-JCP. This is the necessity. □

Let $\Sigma^* = \Sigma(G^*; F^*)$ be a dual polyhedron of $\Sigma = \Sigma(G; F)$. For a circuit C in G, let $C^* = \{e^* |\ \forall e \in C\}$, or say the corresponding vector in $\mathcal{G}_1^*$, of $C \in \mathcal{G}_1$.

Lemma 1 *Let C be a circuit in Σ. Then, $G^*\setminus C^*$ has at most two connected components.*

Proof Suppose H^* be a connected component of $G^*\setminus C^*$ but not the only one. Let D be the subset of F corresponding to $V(H^*)$. Then,

$$C' = \sum_{f\in D} \partial_2 f \subseteq C.$$

However, if $\emptyset \neq C' \subset C$, then C itself is not a circuit. This is a contradiction to the condition of the lemma. From that any edge appears twice in the members of F, there is only one possibility that

$$C = \sum_{f\in F\setminus D} \partial_2 f.$$

Hence, $F\setminus D$ determines the other connected component of $G^*\setminus C^*$ when $C' = C$. □

Any circuit C in G which is the underlying graph of a polyhedron $\Sigma = \Sigma(G; F)$ is said to have the *second Jordan curve property,* or simply write 2-JCP for Σ with its dual $\Sigma^* = \Sigma(G^*; F^*)$ if $G^*\setminus C^*$ has exactly two connected components. A graph G is said to have the 2-JCP if all the circuits in G have the property.

Theorem B(Second Jordan curve theorem) *A graph G has the 2-JCP if, and only if, G is planar.*

Proof To prove the necessity. Because for any circuit C in G, $G^*\setminus C^*$ has exactly two connected components, any C^* which corresponds to a circuit C in G is a cocircuit. Since any edge in G^* appears exactly twice in the elements of V^*, which are all cocircuits, from Lemma 1, V^* contains a basis of Ker δ_1^*. Moreover, V^* is a subset of Im δ_0^*. Hence, Ker $\delta_1 \subseteq$ Im δ_0. From Lemma 4.3.2 in [1], Im $\delta_0^* \subseteq$ Ker δ_1^*. Then, we have Ker δ_1^* =Im δ_0^*, *i.e.*, $\widetilde{\mathcal{H}}_1(\Sigma^*) = 0$. From the dual case of Theorem 4.3.2 in [1], G^* is planar and hence so is G. Conversely, to

prove the sufficiency. From the planar duality, for any circuit C in G, C^* is a cocircuit in G^*. Then, $G^*\backslash C^*$ has two connected components and hence C has the 2- JCP. $\square$

For a graph G, of course connected without loop, associated with a polyhedron $\Sigma = \Sigma(G;F)$, let C be a circuit and E_C, the set of edges incident to, but not on C. We may define an equivalence on E_C, denoted by $\sim_C$ as the transitive closure of that $\forall a, b \in E_C$,

$$\begin{aligned} a \sim_C b \Leftrightarrow \quad & \exists f \in F,\ (a^\alpha C(a,b) b^\beta \subset f) \\ & \vee (b^{-\beta} C(b,a) a^{-\alpha} \subset f), \end{aligned} \tag{C}$$

where $C(a,b)$, or $C(b,a)$ is the common path from a to b, or from b to a in $C \cap f$ respectively. It can be seen that $|E_C/\sim_C| \leqslant 2$ and the equality holds for any C not in F only if Σ is orientable.

In this case, the two equivalent classes are denoted by $E_{\mathcal{L}} = E_{\mathcal{L}}(C)$ and $E_{\mathcal{R}} = E_{\mathcal{R}}(C)$. Further, let $V_{\mathcal{L}}$ and $V_{\mathcal{R}}$ be the subsets of vertices by which a path between the two ends of two edges in $E_{\mathcal{L}}$ and $E_{\mathcal{R}}$ without common vertex with C passes respectively.

From the connectedness of G, it is clear that $V_{\mathcal{L}} \cup V_{\mathcal{R}} = V\backslash V(C)$. If $V_{\mathcal{L}} \cap V_{\mathcal{R}} = \emptyset$, then C is said to have the *third Jordan curve property,* or simply write 3-JCP. In particular, if C has the 3-JCP, then every path from $V_{\mathcal{L}}$ to $V_{\mathcal{R}}$ (or vice versa) crosses C and hence C has the 1-JCP. If every circuit which is not the boundary of a face f of $\Sigma(G)$, one of the underlain polyhedra of G has the 3-JCP, then G is said to have the 3-JCP as well.

Lemma 2 Let C be a circuit of G which is associated with an orientable polyhedron $\Sigma = \Sigma(G;F)$. If C has the 2-JCP, then C has the 3-JCP. Conversely, if $V_{\mathcal{L}}(C) \neq \emptyset,\ V_{\mathcal{R}}(C) \neq \emptyset$ and C has the 3-JCP, then C has the 2-JCP.

Proof For a vertex $v^* \in V^* = V(G^*)$, let $f(v^*) \in F$ be the corresponding face of Σ. Suppose In^* and Ou^* are the two connected components of $G^*\backslash C^*$ by the 2-JCP of C. Then,

$$In = \bigcup_{v^* \in In^*} f(v^*) \text{ and } Ou = \bigcup_{v^* \in Ou^*} f(v^*)$$

are subgraphs of G such that $In \cup Ou = G$ and $In \cap Ou = C$. Also, $E_{\mathcal{L}} \subset In$ and $E_{\mathcal{R}} \subset Ou$ (or vice versa). The only thing remained is to show $V_{\mathcal{L}} \cap V_{\mathcal{R}} = \emptyset$. By contradiction, if $V_{\mathcal{L}} \cap V_{\mathcal{R}} \neq \emptyset$, then In and Ou have a vertex which is not on C in common and hence have an edge incident with the vertex, which is not on C, in common. This is a contradiction to $In \cap Ou = C$.

Conversely, from Lemma 1, we may assume that $G^*\backslash C^*$ is connected by contradiction. Then there exists a path P^* from v_1^* to v_2^* in $G^*\backslash C^*$ such that $V(f(v_1^*)) \cap V_{\mathcal{L}} \neq \emptyset$ and $V(f(v_2^*)) \cap V_{\mathcal{R}} \neq \emptyset$. Consider

$$H = \bigcup_{v^* \in P^*} f(v^*) \subseteq G.$$

Suppose $P = v_1 v_2 \cdots v_l$ is the shortest path in H from $V_{\mathcal{L}}$ to $V_{\mathcal{R}}$.

To show that P does not cross C. By contradiction, assume that v_{i+1} is the first vertex of P crosses C. From the shortestness, v_i is not in $V_{\mathcal{R}}$. Suppose that subpath $v_{i+1} \cdots v_{j-1},\ i+2 \leqslant j < l$, lies on C and that v_j does not lie on C. By the definition of $E_{\mathcal{L}}$, $(v_{j-1}, v_j) \in E_{\mathcal{L}}$ and

hence $v_j \in V_{\mathcal{L}}$. This is a contradiction to the shortestness. However, from that P does not cross C, $V_{\mathcal{L}} \cap V_{\mathcal{R}} \neq \emptyset$. This is a contradiction to the 3-JCP. $\square$

Theorem C(Third Jordan curve theorem) *Let $G = (V, E)$ be with an orientable polyhedron $\Sigma = \Sigma(G; F)$. Then, G has the 3-JCP if, and only if, G is planar.*

Proof From Theorem B and Lemma 2, the sufficiency is obvious. Conversely, assume that G is not planar. By Lemma 4.2.6 in [1], $\mathrm{Im}\partial_2 \subseteq \mathrm{Ker}\partial_1 = \mathcal{C}$, the cycle space of G. By Theorem 4.2.5 in [1], $\mathrm{Im}\partial_2 \subset \mathrm{Ker}\partial_1$. Then, from Theorem B, there exists a circuit $C \in \mathcal{C} \setminus \mathrm{Im}\partial_2$ without the 2-JCP. Moreover, we also have that $V_{\mathcal{L}} \neq \emptyset$ and $V_{\mathcal{R}} \neq \emptyset$. If otherwise $V_{\mathcal{L}} = \emptyset$, let

$$D = \{f | \exists e \in E_{\mathcal{L}}, e \in f\} \subset F.$$

Because $V_{\mathcal{L}} = \emptyset$, any $f \in D$ contains only edges and chords of C, we have

$$C = \sum_{f \in D} \partial_2 f$$

that contradicts to $C \notin \mathrm{Im}\partial_2$. Therefore, from Lemma 2, C does not have the 3-JCP. The necessity holds. $\square$

§2 Reducibilities

For S_g as a surface(orientable, or nonorientable) of genus g, If a graph H is not embedded on a surface S_g but what obtained by deleting an edge from H is embeddable on S_g, then H is said to be *reducible* for S_g. In a graph G, the subgraphs of G homeomorphic to H are called a type of *reducible configuration* of G, or shortly a *reduction*. Robertson and Seymour in [2] has been shown that graphs have their types of reductions for a surface of genus given finite. However, even for projective plane the simplest nonorientable surface, the types of reductions are more than 100 [3,7].

For a surface S_g, $g \geqslant 1$, let $\mathcal{H}_{g-1}$ be the set of all reductions of surface S_{g-1}. For $H \in \mathcal{H}_{g-1}$, assume the embeddings of H on S_g have ϕ faces. If a graph G has a decomposition of ϕ subgraphs H_i, $1 \leqslant i \leqslant \phi$, such that

$$\bigcup_{i=1}^{\phi} H_i = G; \ \bigcup_{i \neq j}^{\phi} (H_i \bigcap H_j) = H; \tag{1}$$

all H_i, $1 \leqslant i \leqslant \phi$, are planar and the common vertices of each H_i with H in the boundary of a face, then G is said to be with the *reducibility* 1 for the surface S_g.

Let $\Sigma^* = (G^*; F^*)$ be a polyhedron which is the dual of the embedding $\Sigma = (G; F)$ of G on surface S_g. For surface S_{g-1}, a reduction $H \subseteq G$ is given. Denote $H^* = [e^* | \forall e \in E(H)]$. Naturally, $G^* - E(H^*)$ has at least $\phi = |F|$ connected components. If exact ϕ components and each component planar with all boundary vertices are successively on the boundary of a face, then Σ is said to be with the *reducibility* 2.

A graph G which has an embedding with reducibility 2 then G is said to be with *reducibility* 2 as well.

Given $\Sigma = (G; F)$ as a polyhedron with under graph $G = (V, E)$ and face set F. Let H be a reduction of surface S_{p-1} and, $H \subseteq G$. Denote by C the set of edges on the boundary of H in G and E_C, the set of all edges of G incident to but not in H. Let us extend the relation $\sim_C$: $\forall a, b \in E_C$,

$$a \sim_C b \Leftrightarrow \exists f \in F_H,\ a, b \in \partial_2 f \tag{2}$$

by transitive law as a equivalence. Naturally, $|E_C / \sim_C| \leqslant \phi_H$. Denote by $\{E_i | 1 \leqslant i \leqslant \phi_C\}$ the set of equivalent classes on E_C. Notice that $E_i = \emptyset$ can be missed without loss of generality. Let V_i, $1 \leqslant i \leqslant \phi_C$, be the set of vertices on a path between two edges of E_i in G avoiding boundary vertices. When $E_i = \emptyset$, $V_i = \emptyset$ is missed as well. By the connectedness of G , it is seen that

$$\bigcup_{i=1}^{\phi_C} V_i = V - V_H. \tag{3}$$

If for any $1 \leqslant i < j \leqslant \phi_C$, $V_i \cap V_j = \emptyset$, and all $[V_i]$ planar with all vertices incident to E_i on the boundary of a face, then H, G as well, is said to be with *reducibility* 3.

§3. Reducibility Theorems

Theorem 1 *A graph G can be embedded on a surface $S_g (g \geqslant 1)$ if, and only if, G is with the reducibility* 1.

Proof Necessity. Let $\mu(G)$ be an embedding of G on surface $S_g (g \geqslant 1)$. If $H \in \mathcal{H}_{g-1}$, then $\mu(H)$ is an embedding on $S_g (g \geqslant 1)$ as well. Assume $\{f_i | 1 \leqslant i \leqslant \phi\}$ is the face set of $\mu(H)$, then $G_i = [\partial f_i + E([f_i]_{in})]$, $1 \leqslant i \leqslant \phi$, provide a decomposition satisfied by (1). Easy to show that all G_i, $1 \leqslant i \leqslant \phi$, are planar. And, all the common edges of G_i and H are successively in a face boundary. Thus, G is with reducibility 1.

Sufficiency. Because of G with reducibility 1, let $H \in \mathcal{H}_{g-1}$, assume the embedding $\mu(H)$ of H on surface S_g has ϕ faces. Let G have ϕ subgraphs H_i, $1 \leqslant i \leqslant \phi$, satisfied by (1), and all H_i planar with all common edges of H_i and H in a face boundary. Denote by $\mu_i(H_i)$ a planar embedding of H_i with one face whose boundary is in a face boundary of $\mu(H)$, $1 \leqslant i \leqslant \phi$. Put each $\mu_i(H_i)$ in the corresponding face of $\mu(H)$, an embedding of G on surface $S_g (g \geqslant 1)$ is then obtained. $\square$

Theorem 2 *A graph G can be embedded on a surface $S_g (g \geqslant 1)$ if, and only if, G is with the reducibility* 2.

Proof Necessity. Let $\mu(G) = \Sigma = (G; F)$ be an embedding of G on surface $S_g (g \geqslant 1)$ and $\mu^*(G) = \mu(G^*) = (G^*, F^*)(= \Sigma^*)$, its dual. Given $H \subseteq G$ as a reduction. From the duality between the two polyhedra $\mu(H)$ and $\mu^*(H)$, the interior domain of a face in $\mu(H)$ has at least a vertex of G^*, $G^* - E(H^*)$ has exactly $\phi = |F_{\mu(H)}|$ connected components. Because of each component on a planar disc with all boundary vertices successively on the boundary of the disc, H is with the reducibility 2. Hence, G has the reducibility 2.

Sufficiency. By employing the embedding $\mu(H)$ of reduction H of G on surface $S_g (g \geqslant 1)$ with reducibility 2, put the planar embedding of the dual of each component of $G^* - E(H^*)$ in

the corresponding face of $\mu(H)$ in agreement with common boundary, an embedding of $\mu(G)$ on surface $S_g(g \geqslant 1)$ is soon done. □

Theorem 3 *A* 3*-connected graph G can be embedded on a surface $S_g(g \geqslant 1)$ if, and only if, G is with reducibility* 3.

Proof Necessity. Assume $\mu(G) = (G, F)$ is an embedding of G on surface $S_g(g \geqslant 1)$. Given $H \subseteq G$ as a reduction of surface S_{p-1}. Because of $H \subseteq G$, the restriction $\mu(H)$ of $\mu(G)$ on H is also an embedding of H on surface $S_g(g \geqslant 1)$. From the 3-connectedness of G, edges incident to a face of $\mu(H)$ are as an equivalent class in E_C. Moreover, the subgraph determined by a class is planar with boundary in coincidence, *i.e.*, H has the reducibility 3. Hence, G has the reducibility 3.

Sufficiency. By employing the embedding $\mu(H)$ of the reduction H in G on surface $S_g(g \geqslant 1)$ with the reducibility 3, put each planar embedding of $[V_i]$ in the interior domain of the corresponding face of $\mu(H)$ in agreement with the boundary condition, an embedding $\mu(G)$ of G on $S_g(g \geqslant 1)$ is extended from $\mu(H)$. □

§4. Research Notes

A. On the basis of Theorems 1–3, the surface embeddability of a graph on a surface(orientabl or nonorientable) of genus smaller can be easily found with better efficiency.

For an example, the sphere S_0 has its reductions in two class described as $K_{3,3}$ and K_5. Based on these, the characterizations for the embeddability of a graph on the torus and the projective plane has been established in [4]. Because of the number of distinct embeddings of K_5 and $K_{3,3}$ on torus and projective plane much smaller as shown in the Appendix of [5], the characterizations can be realized by computers with an algorithm much efficiency compared with the existences, *e.g.*, in [7].

B. The three polyhedral forms of Jordan closed planar curve axiom as shown in section 2 initiated from Chapter 4 of [6] are firstly used for surface embeddings of a graph in [4]. However, characterizations in that paper are with a mistake of missing the boundary conditions as shown in this paper.

C. The condition of 3-connectedness in Theorem 3 is not essential. It is only for the simplicity in description.

D. In all of Theorem 1–3, the conditions on planarity can be replaced by the corresponding Jordan curve property as shown in section 2 as in [4] with the attention of the boundary conditions.

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Math.Combin.Book Ser. Vol.3 (2011), 68-75

Mediate Dominating Graph of a Graph

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Abstract: Let S be the set of minimal dominating sets of graph G and $U,\ W \subset S$ with $U \bigcup W = S$ and $U \bigcap W = \emptyset$. A *Smarandachely mediate-*(U, W) *dominating graph* $D_m^S(G)$ of a graph G is a graph with $V(D_m^S(G)) = V' = V \bigcup U$ and two vertices $u, v \in V'$ are adjacent if they are not adjacent in G or $v = D$ is a minimal dominating set containing u. particularly, if $U = S$ and $W = \emptyset$, i.e., a Smarandachely mediate-$(S, \emptyset)$ dominating graph $D_m^S(G)$ is called the *mediate dominating graph* $D_m(G)$ of a graph G. In this paper, some necessary and sufficient conditions are given for $D_m(G)$ to be connected, Eulerian, complete graph, tree and cycle respectively. It is also shown that a given graph G is a mediate dominating graph $D_m(G)$ of some graph. One related open problem is explored. Finally, some bounds on domination number of $D_m(G)$ are obtained in terms of vertices and edges of G.

Key Words: Connectedness, connectivity, Eulerian, hamiltonian, dominating set, Smarandachely mediate-(U, W) dominating graph.

AMS(2010): 05C69

§1. Introduction

The graphs considered here are finite and simple. Let $G = (V, E)$ be a graph and let the vertices and edges of a graph G be called the elements of G. The undefined terminology and notations can be found in [2]. The connectivity(edge connectivity) of a graph G, denoted by $\kappa(G)(\lambda(G))$, is defined to be the largest integer k for which G is k-connected(k-edge connected). For a vertex v of G, the eccentricity $ecc_G(v)$ of v is the largest distance between v and all the other vertices of G, i.e., $ecc_G(v) = max\{d_G(u, v)/u \in V(G)\}$. The diameter $diam(G)$ of G is the $\max\{ecc_G(v)/v \in V(G)\}$. The chromatic number $\chi(G)$ of a graph G is the minimum number of independent subsets that partition the vertex set of G. Any such minimum partition is called a chromatic partition of $V(G)$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. We call G and H to be isomorphic, and we write $G \cong H$, if there exists a bijection $\theta : V(G) \longrightarrow V(H)$ with $xy \in E(G)$ if and only if $\theta(x)\theta(y) \in E(H)$ for all $x, y \in V(G)$.

[1]Supported by UGC-SAP DRS-II New Delhi, India: for 2010-2015and the University Grants Commission, New Delhi,India-No.F.4-3/2006(BSR)/7-101/2007(BSR) dated: September 2009.

[2]Received April 8, 2011. Accepted August 28, 2011.

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D of G is minimal if for any vertex $v \in D$, $D - v$ is not a dominating set of G. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set of G. The upper domination number $\Gamma(G)$ of G is the maximum cardinality of a minimal dominating set of G. For details on $\gamma(G)$, refer [1].

The maximum number of classes of a domatic partition of G is called the domatic number of G and is denoted by $d(G)$. The vertex independence number $\beta_0(G)$ is the maximum cardinality among the independent set of vertices of G.

Our aim in this paper is to introduce a new graph valued function in the field of domination theory in graphs.

Definition 1.1 *Let S be the set of minimal dominating sets of graph G and U, $W \subset S$ with $U \bigcup W = S$ and $U \bigcap W = \emptyset$. A Smarandachely mediate-(U, W) dominating graph $D_m^S(G)$ of a graph G is a graph with $V(D_m^S(G)) = V' = V \bigcup U$ and two vertices u, $v \in V'$ are adjacent if they are not adjacent in G or $v = D$ is a minimal dominating set containing u. particularly, if $U = S$ and $W = \emptyset$, i.e., a Smarandachely mediate-$(S, \emptyset)$ dominating graph $D_m^S(G)$ is called the mediate dominating graph $D_m(G)$ of a graph G.*

In Fig.1, a graph G and its mediate dominating graph $D_m(G)$ are shown.

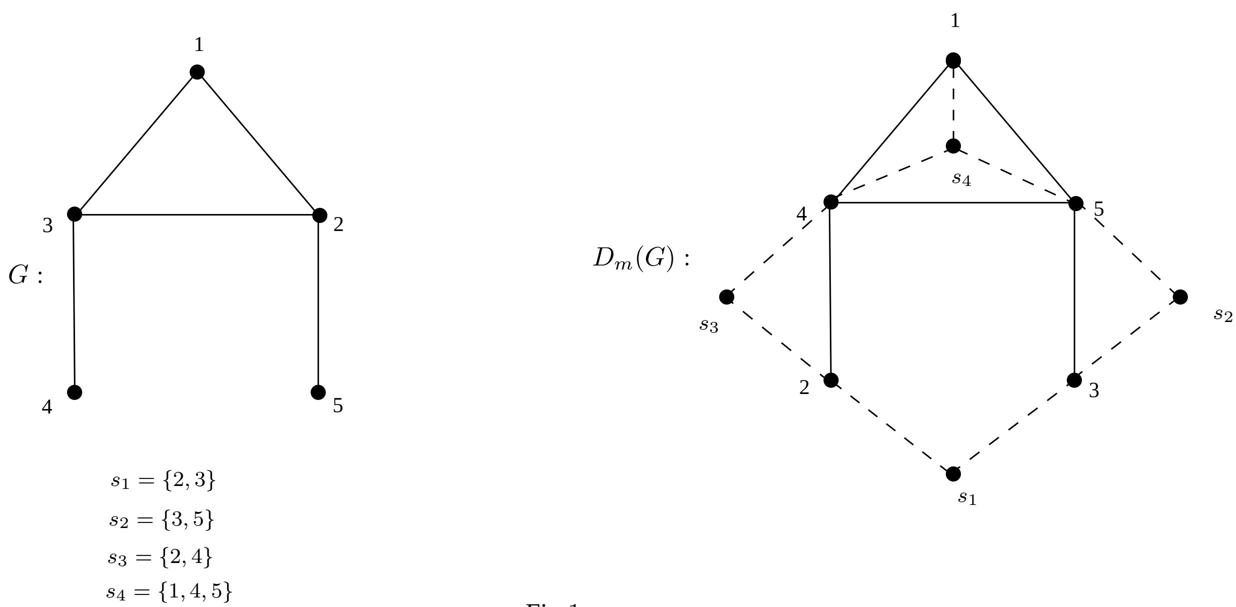

Fig.1

Observations 1.2 The following results are easily observed.:

(1) For any graph G, $\overline{G}$ is an induced subgraph of $D_m(G)$.

(2) Let $S = \{s_1, s_2, \cdots, s_n\}$ be the set of all minimal dominating sets of G, then each s_i; $1 \le i \le n$ will be independent in $D_m(G)$.

(3) If $G = K_p$, then $D_m(G) = pK_2$. (4) If $G = \overline{K}_p$, then $D_m(G) = K_{p+1}$.

§2. Results

When defining any class of graphs, it is desirable to know the number of vertices and edges. It is hard to determine for mediate dominating graph. So we obtain a bounds for $D_m(G)$ to determine the number of vertices and edges in $D_m(G)$.

Theorem 2.1 *For any graph G, $p+d(G) \leq p' \leq \dfrac{p(p+1)}{2}$, where $d(G)$ is the domatic number of G and p' denotes the number of vertices of $D_m(G)$. Further the lower bound is attained if and only if $G=\overline{K}_p$ and the upper bound is attained if and only if G is a $(p-2)$ regular graph.*

Proof The lower bound follows from the fact that every graph has at least $d(G)$ number of minimal dominating sets of G and the upper bound follows from the fact that every vertex is in at most $(p-1)$ minimal dominating sets of G.

Suppose the lower bound is attained. Then every vertex is in exactly one minimal dominating set of G and hence, every minimal dominating set is independent. Further, for any two minimal dominating sets D and D', every vertex in D is adjacent to every vertex in D'.

Suppose the upper bound is attained. Then each vertex is in exactly $(p-1)$ minimal dominating sets hence G is $(p-2)$ regular.

Conversely, we first consider the converse part of the equality of the lower bound. If $G=\overline{K}_p$, then $d(\overline{K}_p)=1$ and there exist exactly one minimal dominating set $S(G)$. Therefore by the definition of $D_m(G)$, $V(D_m(G))=p+|S(G)|=p+1=p+d(G)$.

Now, we consider the converse part of the equality of the upper bound. Suppose G is a $(p-2)$ regular graph. Then G has $\dfrac{p(p-1)}{2}$ minimal dominating sets of G. Therefore by the definition of $D_m(G)$, $V(D_m(G))=p+|S(G)|=p+\dfrac{p(p-1)}{2}=\dfrac{p(p+1)}{2}$. □

Theorem 2.2 *For any graph G, $p \leq q' \leq \dfrac{p(p+1)}{2}$, where q' denotes the number of edges of $D_m(G)$. Further, the lower bound is attained if and only if $G=K_p$ and the upper bound is attained if and only if $G=\overline{K}_p$.*

Proof First we consider the lower bound. Suppose the lower bound is attained. Then $p=q'$, it follows that $\overline{G}$ contains no edges in $D_m(G)$. Therefore by observation 3, $G=K_p$; $p\geq 2$. Conversely, if $G=K_p$; $p \geq 2$ the $D_m(G)=pK_2$. Therefore $p=q'$.

Now consider the upper bound. Suppose the upper bound is attained. Then $q'=\dfrac{p(p+1)}{2}$. Therefore $\delta(D_m(G))=\Delta(D_m(G))=p-1$. Hence $D_m(G)=K_{p+1}$. By observation 4, $G=\overline{K}_p$.

Conversely, if $G=\overline{K}_p$, then $D_m(G)=K_{p+1}$, since K_{p+1} has $\frac{p(p+1)}{2}$ edges. Therefore $q'=\dfrac{p(p+1)}{2}$. □

In the next theorem, we prove the necessary and sufficient condition for $D_m(G)$ to be connected.

Theorem 2.3 *For any (p,q) graph G, the mediate dominating graph $D_m(G)$ is connected if and only if $\Delta(G)<p-1$.*

Proof Let $\Delta(G) < p - 1$. We consider the following cases.

Case 1 Let u and v be any two adjacent vertices in G. Suppose there is no minimal dominating set containing both u and v. Then there exist another vertex w in V which is not adjacent to both u and v. Let D and D' be any two maximal independent sets containing u, w and v, w respectively. Since every maximal independent set is a minimal dominating set, hence u and v are connected by a path $uDwD'v$. Thus $D_m(G)$ is connected.

Case 2 Let u and v be any two nonadjacent vertices in G. Then by observation 1, $\overline{G}$ is an induced subgraph of $D_m(G)$. Clearly u and v are connected in $D_m(G)$. Thus from the above two cases $D_m(G)$ is connected.

Conversely, suppose $D_m(G)$ is connected. On the contrary assume that $\Delta(G) = p - 1$. Let u be any vertex of degree $p - 1$. Then u is a minimal dominating set of G and $V - u$ also contains a minimal dominating set of G. It follows that $D_m(G)$ has two components, a contradiction. □

Theorem 2.4 *For any graph G, $D_m(G)$ is either connected or has at least one component which is K_2.*

Proof We consider the following cases:

Case 1 If $\Delta(G) < p - 1$, then by Theorem 2.1, $D_m(G)$ is connected.

Case 2 If $\delta(G) = \Delta(G) = p - 1$, then G is K_p. By Observation 3, $D_m(K_p) = pK_2$.

Case 3 If $\delta(G) < \Delta(G) = p - 1$.

Let $u_1, u_2, \cdots, u_i$ be the vertices of degree $p - 1$ in G. Let $H = G - \{u_1, u_2, \cdots, u_i\}$. Then clearly $\Delta(H) < p - 1$. By Theorem 2.1, $D_m(H)$ is connected. Since $D_m(G) = D_m(H) \cup (\{u_1\} + u_1) \cup (\{u_2\} + u_2) \cup \cdots \cup (\{u_n\} + u_n)$. Therefore it follows that at least one component of $D_m(G)$ is K_2. □

Corollary 1 *For any graph G, $D_m(G) = K_p \cup K_2$ if and only if $G = K_{1,p-1}$.*

Proof The proof follows from Observation 3 and Theorem 2.6. □

In the next theorem, we characterize the graphs G for which $D_m(G)$ is a tree.

Theorem 2.5 *The mediate dominating graph $D_m(G)$ of G is a tree if and only if $G = K_1$.*

Proof Let the mediate dominating graph $D_m(G)$ of G be a tree and $G \neq K_1$. Then by Theorem 2.3, $\Delta(G) < p - 1$. Hence $D_m(G)$ is connected. Now consider the following cases.

Case 1 Let G be a disconnected graph. If G is totally disconnected graph, then by the observation 4, $D_m(G) = K_{p+1}$, a contradiction.

Let us consider at least one component of G containing an edge uev. Then the smallest possible graph is $G = K_2 \cup K_1$. Therefore $D_m(G) = C_3 \cdot C_3$, a contradiction. Hence for any disconnected graph G of order at least two, $D_m(G)$ must contain a cycle of length at least three,

a contradiction. Thus $G = K_1$.

Case 2 Let G be a connected graph with $\Delta(G) < p-1$. By Theorem 2.3, $D_m(G)$ is connected. For $D_m(G)$ to be connected and $\Delta(G) < p-1$, the order of the graph G must be greater than or equal to four. Then there exist at least two nonadjacent vertices u and v in G, which belong to at least one minimal dominating set D of G. Therefore $uvDu$ is a cycle in $D_m(G)$, a contradiction. Thus from above two cases we conclude that $G = K_1$.

Conversely, if $G = K_1$, then by the definition of $D_m(G)$, $D_m(G) = K_2$, which is a tree. □

In the next theorem we characterize the graphs G for which $D_m(G)$ is a cycle.

Theorem 2.6 *The mediate dominating graph $D_m(G)$ of G is a cycle if and only if $G = 2K_1$.*

Proof Let $D_m(G)$ be a cycle. Then by Theorem 2.3, $\Delta(G) < p-1$. Suppose $G \neq 2K_1$, then by Theorem 2.5, $D_m(G)$, $D_m(G)$ is either a tree or containing at least one vertex of degree greater than or equal to 3, a contradiction. Hence $G = 2K_1$.

Conversely, if $G = 2K_1$ then by observation, $D_m(G) = K_3$ or C_3 a cycle. □

Proposition 1 *The mediate dominating graph $D_m(G)$ of G is a complete graph if and only if $G = \overline{K}_p$.*

In the next theorem, we find the diameter of $D_m(G)$.

Theorem 2.7 *Let G be any graph with $\Delta(G) < p-1$, then $diam(D_m(G)) \leq 3$, where $diam(G)$ is the diameter of G.*

Proof Let G be any graph with $\Delta(G) < p-1$, then by Theorem 2.3, $D_m(G)$ is connected. Let $u, v \in V(D_m(G))$ be any two arbitrary vertices in $D_m(G)$. We consider the following cases.

Case 1 Suppose $u, v \in V(G)$, u and v are nonadjacent vertices in G, then $d_{D_m(G)}(u, v) = 1$. If u and v are adjacent in G, suppose there is no minimal dominating set containing both u and v. Then there exist another vertex w in $V(G)$, which is not adjacent to both u and v. Let D and D' be any two maximal independent sets containing u, w and v, w respectively. Since every maximal independent set is a minimal dominating set, hence u and v are connected in $D_m(G)$ by a path $uDwD'v$. Thus, $d_{D_m(G)}(u, v) \leq 3$.

Case 2 Suppose $u \in V$ and $v \notin V$. Then $v = D$ is a minimal dominating set of G. If $u \in D$, then $d_{D_m(G)}(u, v) = 1$. If $u \notin D$, then there exist a vertex $w \in D$ which is adjacent to both u and v. Hence $d_{D_m(G)}(u, v) = d(u, w) + d(w, v) = 2$.

Case 3 Suppose $u, v \in V$. Then $u = D$ and $v = D'$ are two minimal dominating sets of G. If D and D' are disjoint, then every vertex in $w \in D$ is adjacent to some vertex $x \in D'$ and vice versa. This implies that, $d_{D_m(G)}(u, v) = d(u, w) + d(w, x) + d(x, v) = 3$. If D and D' have a vertex in common, then $d_{D_m(G)}(u, v) = d(u, w) + d(w, v) = 2$. Thus from all these cases the result follows. □

In the next two results we prove the vertex and edge connectivity of $D_m(G)$.

Theorem 2.8 *For any graph G,*

$$\kappa(D_m(G)) = \min\{\min(deg_{D_m(G)} v_i)_{1\le i\le p}, \min_{1\le j\le n} |S_j|\},$$

where $S_j's$ are the minimal dominating sets of G

Proof Let G be a (p,q) graph. We consider the following cases:

Case 1 Let $x \in v_i$ for some i, having minimum degree among all $v_i's$ in $D_m(G)$. If the degree of x is less than any vertex in $D_m(G)$, then by deleting those vertices of $D_m(G)$ which are adjacent with x, results in a disconnected graph.

Case 2 Let $y \in S_j$ for some j, having minimum degree among all vertices of $S_j's$. If degree of y is less than any other vertices in $D_m(G)$, then by deleting those vertices which are adjacent with y, results in a disconnected graph.
Hence the result follows. □

Theorem 2.9 *For any graph G,*

$$\lambda(D_m(G)) = \min\{\min(deg_{D_m(G)} v_i)_{1\le i\le p}, \min_{1\le j\le n} |S_j|\},$$

where $S_j's$ are the minimal dominating sets of G

Proof The proof is on the same lines of the proof of Theorem 2.8. □

§3. Traversability in $D_m(G)$

The following will be useful in the proof of our results.

Theorem A([2]) *A graph G is Eulerian if and only if every vertex of G has even degree. Next, we prove the necessary and sufficient conditions for $D_m(G)$ to be Eulerian.*

Theorem 3.1 *For any graph G with $\Delta(G) < p-1$, D_m is Eulerian if and only if it satisfies the following conditions:*

(i) *Every minimal dominating set contains even number of vertices;*

(ii) *If $v \in V$ is a vertex of odd degree, then it is in odd number of minimal dominating sets, otherwise it is in even number of minimal dominating sets.*

Proof Suppose $\Delta(G) < p-1$. By Theorem 2.3, $D_m(G)$ is connected. If $D_m(G)$ is Eulerian. On the contrary, if condition (i) is not satisfied, then there exists a minimal dominating set containing odd number of vertices and hence $D_m(G)$ has a vertex of odd degree, therefore by Theorem A, $D_m(G)$ is Eulerian, a contradiction. Similarly we can prove (ii). Conversely, suppose the given conditions are satisfied. Then degree of each vertex in $D_m(G)$ is even. Therefore by Theorem A, $D_m(G)$ is Eulerian. □

Theorem 3.2 *Let G be any graph with $\Delta(G) < p-1$ and $\Gamma(G) = 2$. If every vertex is in exactly two minimal dominating sets of G, then $D_m(G)$ is Hamiltonian.*

Proof Let $\Delta(G) < p-1$. Then by Theorem 2.3, $D_m(G)$ is connected. Clearly $\gamma(G) = \Gamma(G)$ and if every vertex is in exactly two minimal dominating sets then there exist an induced two regular graph in $D_m(G)$. Hence $D_m(G)$ contains a hamiltonian cycle. Therefore $D_m(G)$ is hamiltonian. □

Next, we prove the chromatic number of $D_m(G)$.

Theorem 3.3 *For any graph G,*

$$\chi(D_m(G)) = \begin{cases} \chi(\overline{G}) + 1 & \text{if vertices of any minimal dominating sets colored by } \chi(\overline{G}) \text{ colors} \\ \chi(\overline{G}) & \text{otherwise} \end{cases}$$

Proof Let G be a graph with $\chi(\overline{G}) = k$ and D be the set of all minimal dominating sets of G. Since by the definition of $D_m(G)$, $\overline{G}$ is an induced subgraph of $D_m(G)$ and by Observation 2, D is an independent set. Therefore to color $D_m(G)$, either we can make use of the colors which are used to color $\overline{G}$ that is $\chi(D_m(G)) = k = \chi(\overline{G})$ or we should have to use one more new color. In particular, if the vertices of any minimal dominating set x of G are colored with $k-$colors, then we require one more new color to color x in $D_m(G)$. Hence in this case we require $k+1$ colors to color $D_m(G)$. Therefore $\chi(D_m(G)) = k+1$ This implies, $\chi(D_m(G)) = \chi(\overline{G}) + 1$. □

§4. Characterization of $D_m(G)$

Question. *Is it possible to determine the given graph G is a mediate dominating graph of some graph?*

A partial solution to the above problem is as follows.

Theorem 4.1 *If $G = K_p$; $p \geq 2$, then it is a mediate dominating graph of $\overline{K}_{p-1}$.*

Proof The proof follows from Theorem 2.2. □

Problem 4.1 *Give necessary and sufficient condition for a given graph G is a mediate dominating graph of some graph.*

§5. Domination in $D_m(G)$

We first calculate the domination number of $D_m(G)$ of some standard class of graphs.

Theorem 5.1 (*i*) *If $G = K_p$, then $\gamma(D_m(K_p)) = p$;*

(*ii*) *If $G = K_{1,p}$, then $\gamma(D_m(K_{1,p})) = 2$;*

(*iii*) *If $G = W_p$; $p \geq 4$ then $\gamma(D_m(W_p)) = \gamma(\overline{C}_{p-1}) + 1$;*

(*iv*) *If $G = P_p$; $p \geq 2$ then $\gamma(D_m(P_p)) = 2$;* □

Theorem 5.2 *Let G be any graph of order p and $S = \{s_1, s_2, \cdots, s_n\}$ be the set of all minimal dominating sets of G, then $\gamma(D_m(G)) \leq \gamma(\overline{G}) + |S|$.*

Proof Let $D = \{v_1, v_2, \cdots, v_i\}$; $1 \leq i \leq p$ be a minimum dominating set of $\overline{G}$. By the definition of $D_{m(G)}$, $\overline{G}$ is an induced subgraph of $D_m(G)$ and by Observation 2, each s_i; $1 \leq i \leq n$ is independent in $D_m(G)$. Hence $D' = D \cup S$ will form a dominating set in $D_m(G)$. Therefore $\gamma(D_m(G)) \leq |D'| = |D \cup S| = \gamma(\overline{G}) + |S|$. □

Theorem 5.3 *Let G be any connected graph with $\delta(G) = 1$, then $\gamma(D_m(G)) = 2$.*

Proof Let G be any connected graph with a minimum degree vertex u, such that $deg(u) = 1$. Let v be a vertex adjacent to u in G. Then $deg_{\overline{G}}(u) = p - 2$, and every minimal dominating set contains either u or v. Hence $D = \{u, v\}$ is a minimal dominating set of $D_m(G)$. Therefore, $\gamma(D_m(G)) = |D| = |\{u, v\}| = 2$. □

Corollary 2 *For any nontrivial tree T, $\gamma(D_m(T)) = 2$.*

Furthermore, we get a Nordhaus-Gaddum type result following.

Theorem 5.4 *Let G be any graph of order p, then*

(*i*) $\gamma(D_m(G)) + \gamma(D_m(\overline{G})) \leq p + 1$;
(*ii*) $\gamma(D_m(G)) \cdot \gamma(D_m(\overline{G})) \leq p$.

Further, equality holds if and only if $G = K_p$.

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Math.Combin.Book Ser. Vol.3 (2011), 76-81

Graph Theoretic Parameters Applicable to Social Networks

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Abstract: Let $G = (V, E)$ be a graph. When G is used to model the network of a group of individuals, the vertex set V stands for individuals and the edge set E is used to represent the relations between them. If we want a set of representatives having relations with other members of the group, choose a dominating set of the graph. For a smallest set of representatives, choose a minimal dominating set of the graph. In this paper we generalize this concept by allowing the division of the group into a number of subgroups. We introduce the concept of class domination (greed domination) and study it's properties. A dominating set S of G is a class dominating set or a greed dominating set, if $S \bigcap V_i \neq \phi$ for all i. Here V_i such that $i = 1, 2, \ldots, n$ is a partition of V. We also discuss different versions of domination in the context of social networks.

Key Words: Minimal dominating set, greed dominating set, minimal greed dominating set, proportionate greed dominating set.

AMS(2010): 05C69

§1. Introduction

A graph $G = (V, E)$ is a discrete mathematical structure which contains the nonempty set V of vertices and the set E of unordered pairs of elements of V called edges. In this paper we restrict our attention to finite simple graphs. For basic terminology and definitions which are not explained in this paper, reader may refer Harary [4].

Graph is an efficient tool for modeling group of individuals (represented by vertices) and various relationships among them (represented by edges). Consider the problem of selecting representatives from the group, who have good relationship with the remaining members of the group. A dominating set of the graph which model the problem is the solution. The *dominating set* (DS) of a graph $G = (V, E)$ is a subset S of V such that all vertices in $V - S$ is adjacent to at least one vertex in S. A *minimal dominating set* (MDS) is a dominating set S such that $S - \{v\}$ is not a dominating set for all vertex $v \in S$. The *domination number* $\gamma(G)$ and the

[1]Support by sanctioning a minor project No : MRP (S) - 653 / 2007 (X Plan) / KLKE 018 / UGC - SWRO.
[2]Received November 1, 2010. Accepted August 30, 2011.

upper domination number $\Gamma(G)$ of the graph G are defined as follows.

$$\gamma(G) = \min \{|S| : \text{ S is a minimal dominating set of } G\}$$

and

$$\Gamma(G) = \max \{|S| : \text{ S is a minimal dominating set of } G\}.$$

Although the mathematical study of dominating sets in graphs began around 1960, the subject has historical root dating back to 1862 when de Jaenisch [3] studied the problem of determining the minimum number of queens which are necessary to cover an $n \times n$ chessboard. In 1958 Claud Berg [1] wrote a book on graph theory, in which he defined for the first time the concept of domination number of a graph (he called the number, the *coefficient of external stability*). In 1977 Cockayne and Hedetniemi [2] published a survey of the few results known at that time about the dominating sets in graphs. Later the subject has developed as an important area of research with many related areas such as independence, irredundance, packing, covering etc. A comprehensive text on domination is available, which is edited by T. W. Haynes et al. [5]. For advanced research topics, reader may refer another text edited by T. W. Haynes et al. [6].

§2. Greed Domination

A group of people contains Hindus, Christians and Muslims. It is possible that a member of a particular religion has good relation with members of other religion. As a consequence, if we select a minimal set of representatives having good relationship with all other members of the group, the representatives may not contain members from some religion. This situation results into imbalance of social relations. A possible solution is to give due consideration to all subgroups while selecting the representatives. This motivates us to generalize the concept of dominating sets in graphs.

Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \ldots V_n\}$ be a mutually disjoint partition of V. Total number of subsets in the partition P is denoted by $|P|$. A subset S of V is called a *greed dominating set (class dominating set)* of G w.r.t to the partition P, if S dominate all vertices of $V - S$ and $V_i \cap S \neq \phi$ for all $i = 1, 2, \ldots, n$. A greed dominating set S is a *minimal greed dominating set* if no proper subset of S is a greed dominating set. The *greed domination number* $\gamma_{gP}(G)$ and the *upper greed domination number* $\Gamma_{gP}(G)$ of the graph G are defined as follows.

$$\gamma_{gP}(G) = \min \{|S| : \text{ S is a minimal greed dominating set of } G\}$$

and

$$\Gamma_{gP}(G) = \max \{|S| : \text{ S is a minimal greed dominating set of } G\}.$$

When $P = \{V\}$, greed domination coincides with ordinary domination. For any partition P of V, at least one minimal greed dominating set exists. Hence the definitions of $\gamma_{gP}(G)$ and $\Gamma_{gP}(G)$ are meaningful. Let P_1 and P_2 are two partitions of V. We say that P_2 is bigger than P_1 or P_1 is smaller than P_2 if P_2 is obtained by further partitioning one or more subsets of P_1. Two partitions P_1 and P_2 are incomparable, if P_2 is not bigger than P_1 or vice versa.

Theorem 2.1 *If P is a partition of G such that $|P| = n$, then $\gamma_{gP}(G) \geq n$.*

Proof Any minimal dominating set S of G w.r.t the partition $P = \{V_1, V_2, \ldots, V_n\}$ satisfies $S \cap V_i \neq \phi$ for all $i = 1, 2, \ldots, n$. Hence $|S| \geq n$ and $\gamma_{gP}(G) \geq n$. □

Is it possible that for partition P, $\gamma_{gP}(G) > |P|$? The answer is YES. It is illustrated below.

Example 2.2 Consider the graph G, which is the union of the cycles (v_1, v_2, v_3), (v_6, v_7, v_8) and the path (v_3, v_4, v_5, v_6). Clearly $\gamma = 2$. Consider the partition $P = \{V_1, V_2\}$ of V such that, $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V_2 = \{v_7, v_8\}$. For this partition, $\gamma_{gP}(G) = 3 > |P|$.

Theorem 2.3 *If P is a partition such that, $\gamma_{gP}(G) = |P|$, and for any partition P' where P' is bigger than P, obtained by partitioning exactly one subset of P and $|P'| = |P| + 1$, then $\gamma_{gP'}(G) = |P| + 1$.*

Proof Let $S = \{v_1, v_2, \ldots, v_{|P|}\}$ be a minimal greed dominating set of G w.r.t $P = \{V_1, V_2, \ldots, V_{|P|}\}$ such that $v_i \in V_i$ for $i = 1, 2, \ldots, |P|$. Let P' be obtained by further partitioning exactly one of the subsets, say V_1 into to subsets V_{11} and V_{12}. If $v_1 \in V_{11}$ then $v_1 \notin V_{12}$ and vice versa. For the time being let $v_1 \in V_{11}$. Now consider $S' = \{v, v_1, v_2 \ldots, v_{|P|}\}$, where $v \in V_{12}$. Clearly S' is a minimal greed dominating set of G w.r.t the new partition P'. Hence the result. □

Corollary 2.4 *If $P_1, P_2, \ldots, P_n$ are partitions of $V(G)$ satisfying the conditions,*

(i) *P_{i+1} is bigger than P_i;*
(ii) *$|P_{i+1}| = |P_i| + 1$ for each i;*
(iii) *$\gamma_{gP_1}(G) = |P_1|$,*

then $\gamma_{gP_{i+1}}(G) = \gamma_{gP_i}(G) + 1$ for each i.

Next we shall characterize the graphs such that $\gamma_{gP}(G) = |P|$ for each partition P of $V(G)$.

Theorem 2.5 *For the graph G, $\gamma_{gP}(G) = |P|$ for all partition P of $V(G)$ if and only if there exists a vertex $v \in V$ such that $N[v] = V(G)$.*

Proof Suppose the graph G has the property, $\gamma_{gP}(G) = |P|$ for for each partition P of $V(G)$. Consider the partition $P = \{V\}$. Then $\gamma_{gP}(G) = 1$. Hence there exists a vertex $v \in V$ such that $N[v] = V(G)$.

Conversely, Let there exists a vertex $v \in V$ such that $N[v] = V(G)$. Take any partition $P = \{V_1, V_2, \ldots, V_n\}$ of $V(G)$. With no loss of generality we can assume that, $v \in V_1$. Now consider the set $S = \{v, v_2, v_3, \ldots, v_n\}$ made by selecting v from V_1 and an arbitrary vertex v_i from V_i for $i = 2, 3, \ldots, n$. This set is a minimal greed dominating set of G w.r.t the partition P. Hence $\gamma_{gP}(G) = |P|$, by Theorem 2.1. □

Theorem 2.6 *Let P_1 and P_2 are two partitions of V such that P_2 is bigger than P_1, then $\gamma_{gP_1}(G) \leq \gamma_{gP_2}(G)$.*

Proof Suppose that S is a minimal greed dominating set of the graph G w.r.t the partition P_2 such that $\gamma_{gP_2}(G) = |S|$. Then S is a greed dominating set of G w.r.t the partition P_1. Hence $\gamma_{gP_1}(G) \leq \gamma_{gP_2}(G) = |S|$. □

Theorem 2.7 *If γ is the domination number of the graph G, then $V(G)$ has a partition P such that $\gamma_{gP}(G) = \gamma$.*

Proof Let $S = \{v_1, v_2, \ldots, v_\gamma\}$ be a minimal dominating set of G. Consider the partition $P = \{V_1, V_2, \ldots, V_\gamma\}$ of V such that $v_i \in V_i$ for all $i = 1, 2, \ldots, \gamma$. Now $\gamma_{gP}(G) = \gamma$. □

Theorem 2.8 *If P is a partition such that $\gamma_{gP}(G) = \gamma$, then $\gamma_{gP'}(G) = \gamma$ for all partition P' smaller than P.*

Proof Let P' be smaller than P. Then P' is obtained by combining two or more subsets of P. Suppose S' is the smallest minimal greed dominating set of G w.r.t the partition P' and $|S| > \gamma$. Since $\gamma_{gP}(G) = \gamma$, there exists a minimal greed dominating set S w.r.t P such that $|S| = \gamma$. But intersection of S with any subset of P' is nonempty. This gives another minimal greed dominating set of G w.r.t P'. Also $|S| < |S'|$. This is a contradiction. □

§3. Proportionate Greed Domination

A greed dominating set S of the graph G is called a *proportionate greed dominating set* (PGDS) w.r.t. the partition $P = \{V_1, V_2, \ldots, V_n\}$, if $\dfrac{|S \cap Vi|}{|V_i|} = \dfrac{|S \cap Vj|}{|V_j|}$ for all $i, j = 1, 2, \ldots, n$. This idea is a special case of the concept of greed dominating set. A proportionate greed dominating set S is called a minimal proportionate greed dominating set (MPGDS) if no proper subset of S is a proportionate greed dominating set. MPGDS is used to model the problem of selecting representatives from a group of individuals, so that the number of representatives is proportionate to the strength of the subgroups.

Theorem 3.1 *The graph $G = (V, E)$ has a PGDS w.r.t the partition P where $|P| \neq |V|$ if and only if $|V|$ is not a prime number.*

Proof Let S be a PGDS w.r.t the partition $P = \{V_1, V_2, \ldots, V_n\}$ of the graph G. Then by definition of PGDS, $\dfrac{|S \cap Vi|}{|V_i|} = \dfrac{|S \cap Vj|}{|V_j|} = \dfrac{p}{q}$ for all $i, j = 1, 2, \ldots, n$, where p and q are relatively prime positive integers and $q \neq 0$. Clearly, q divides $|S \cap Vi|$ and p divides $|Vi|$ for all i. Then $|V| = \sum_i |V_i|$ is divisible by p. If $p = 1$, then $|V_i| = q \times |S \cap Vi|$ for all i. Now $|V|$ is divisible by q. Hence always $|V|$ is not a prime number.

Conversely, let $|V| = qr$, where $q, r > 1$ and $P = \{V_1, V_2, \ldots, V_n\}$ be a partition of V such that $|V_i| = qr_i$ for all i and $\sum_i r_i = r$. Then the set $S = V$ itself is a PGDS of G w.r.t the given partition. □

If a graph has a PGDS w.r.t. a partition P, then it has an MPGDS. This fact leads to the following result.

Corollary 2.2 *The graph $G = (V, E)$ has an MPGDS w.r.t the partition P where $|P| \neq |V|$ if and only if $|V|$ is not a prime number.*

Theorem 3.3 *If S is a PGDS w.r.t the partition $P = \{V_1, V_2, \ldots, V_n\}$ of the graph G, then $\frac{|S \cap Vi|}{|V_i|} = \frac{|S|}{|V|} = \frac{p}{q}$ for all $i = 1, 2, \ldots, n$.*

Proof Since $\frac{|S \cap Vi|}{|V_i|} = \frac{|S|}{|V|} = \frac{p}{q}$ for all i and $(p, q) = 1$, $|S \cap Vi| = n_i p$ and $|V_i| = n_i q$ where n_i is some positive integer. Then $|S| = \sum_i |S \cap Vi| = \sum_i n_i p$ and $|V| = \sum_i |V_i| = \sum_i n_i q$. Hence the result. □

But in the graphs modeling real situations we cannot ensure the equality of the fractions $\frac{|S \cap Vi|}{|V_i|}$. To deal with these cases we allow variations of the values $\frac{|S \cap Vi|}{|V_i|}$, subject to the condition $|\frac{p}{q} - \frac{|S \cap Vi|}{|V_i|}| \leq \epsilon$, where ϵ has a prescribed value. Using Theorem 3.3 we get an approximate value of $\frac{|S \cap Vi|}{|V_i|}$ for graphs having no PGDS w.r.t the partition P.

§4. Cost Factor of a Partition

If the graph G models a set of people, then $\gamma(G)$ is the minimum number of representatives selected from the group. But in many situations, where considerations of group within group is strong, this is not practical. Consequently selection of more representatives than the minimum required increases the total cost. Another interesting situation arise while establishing communication networks. If radio stations are to be situated at different places in a country, naturally we select those places such that every part of the country receive signals from at least one station. To minimize the total cost, we try to minimize the number of places selected. Then some states may not get a radio station. To solve this problem, every state is given minimum one radio station, which undermines our objective. Keeping this fact in mind we introduce the *cost factor* of the partition P. The cost factor of the partition P is defined as $C_P(G) = \gamma_{gP}(G) - \gamma(G)$. A partition P of $V(G)$ is called a *cost effective partition* if $C_P(G) = 0$. Every graph has at least one cost effective partition.

Theorem 4.1 *Let $G = (V, E)$ be a graph, then*

(i) *G has at least one cost effective partition;*

(ii) *G has exactly one cost effective partition if and only if $\gamma(G) = |V|$.*

Proof The conclusion (i) follows from Theorem 2.7. For (ii), if $\gamma(G) = |V|$ and $P = \{V_1, V_2, \ldots, V_{|V|}\}$ is a partition of V, then $|V_i| = 1$ for each i. If there exists another partition P' such that $|P'| = |V|$, then $P = P'$.

To prove converse part, Let the graph G has exactly one cost effective partition, say $P = \{V_1, V_2, \ldots, V_\gamma\}$. Suppose $\gamma(G) < |V|$. Since P is cost effective, $\gamma_{gP}(G) = \gamma(G)$ and let S be the corresponding greed dominating set. Take the vertex $v \in (V - S)$. If necessary

rename the subset of the partition such that, $v \in V_1$. Next consider the new partition $P' = \{V_1 - \{v\}, V_2 \cup \{v\}, V_3, \ldots, V_\gamma\}$. Clearly $|P| = |P'|$ and $\gamma_{gP'}(G) = \gamma(G)$. This contradicts the uniqueness of P. □

§5. Problems for Further Research

Here we present a set of questions which are intended for future research.

(i) We have proved in Theorem 2.6 that, for the partitions P_1 and P_2 of V such that P_2 bigger than P_1, $\gamma_{gP_1}(G) \leq \gamma_{gP_2}(G)$. Is there any relation between $\Gamma_{gP_1}(G)$ and $\Gamma_{gP_2}(G)$?

(ii) Is it possible to characterize the partitions of a graph, so that $\gamma_{gP}(G) = |P|$?

(iii) Find the total number of different partitions of the graph G having domination number γ, such that $\gamma_{gP}(G) = \gamma$.

(iv) The subset S of $V(G)$ is a total dominating set, if every vertex in V is adjacent to at least one vertex in S. Extend the idea of greed domination to total dominating sets of G.

(v) Design an algorithm for computing the values of $\gamma_{gP}(G)$ and $\Gamma_{gP}(G)$ for a given partition P of the graph G.

(vi) Find the total number of cost effective partitions of a given graph with n vertices and having domination number γ.

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Math.Combin.Book Ser. Vol.3 (2011), 82-87

Forcing (G,D)-number of a Graph

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Abstract: In [7], we introduced the new concept (G,D)-set of graphs. Let $G = (V, E)$ be any graph. A (G,D)-set of a graph G is a subset S of vertices of G which is both a dominating and geodominating(or geodetic) set of G. The minimum cardinality of all (G,D)-sets of G is called the (G,D)-number of G and is denoted by $\gamma_G(G)$. In this paper, we introduce a new parameter called forcing (G,D)-number of a graph G. Let S be a γ_G-set of G. A subset T of S is said to be a forcing subset for S if S is the unique γ_G-set of G containing T. A forcing subset T of S of minimum cardinality is called a minimum forcing subset of S. The forcing (G,D)-number of S denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of S. The forcing (G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all γ_G-sets S of G and it is denoted by $f_{G,D}(S)$.

Key Words: (G,D)-number, Forcing (G,D)-number, Smarandachely k-dominating set.

AMS(2010): 05C69

§1. Introduction

By a graph G=(V,E), we mean a finite, undirected connected graph without loops and multiple edges. For graph theoretic terminology, we refer [5]. A set of vertices S in a graph G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S. Particularly, if $k = 1$, such a set is called a dominating set of G, i.e., every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number $\gamma(G)$ of G[6]. A u-v geodesic is a u-v path of length d(u,v). A set S of vertices of G is a geodominating (or geodetic) set of G if every vertex of G lies on an x-y geodesic for some x,y in S. The minimum cardinality of a geodominating set is the geodomination (or geodetic) number of G and it is denoted by g(G)[1[-[4]. A (G,D)-set of G is a subset S of V(G) which is both a dominating and geodetic set of G. The minimum cardinality of all (G,D)-sets of G is called the (G,D)-number of G and is denoted by γ_G(G).

[1]Received January 21, 2011. Accepted August 30, 2011.

Any (G,D)-set of G of cardinality γ_G is called a γ_G-set of G[7].In this paper, we introduce a new parameter called forcing (G,D)-number of a graph G. Let S be a γ_G-set of G. A subset T of S is said to be a forcing subset for S if S is the unique γ_G-set of G containing T. A forcing subset T of S of minimum cardinality is called a minimum forcing subset of S. The forcing (G,D)-number of S denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of S. The forcing (G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all γ_G-sets S of G and it is denoted by $f_{G,D}$(S).

§2. Forcing (G,D)-number

Definition 2.1 *Let G be a connected graph and S be a γ_G-set of G. A subset T of S is called a forcing subset for S if S is the unique γ_G-set of G containing T. A forcing subset T of S of minimum cardinality is called a minimum forcing subset for S. The forcing (G,D)-number of S denoted by $f_{G,D}(S)$ is the cardinality of a minimum forcing subset of S. The forcing (G,D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all γ_G-sets S of G and it is denoted by $f_{G,D}(G)$. That is, $f_{G,D}(G) = \min\{f_{G,D}(S)$: S is any γ_G-set of G$\}$.*

Example 2.2 In the following figure,

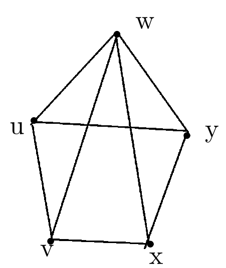

Fig.2.1

$S_1 = \{u, x\}$ and $S_2 = \{v, y\}$ are the only two γ_G-sets of G. $\{u\}, \{x\}$ and $\{u, x\}$ are forcing subsets of S_1. Therefore, $f_{G,D}(S_1) = 1$. Similarly, $\{v\}$, $\{y\}$ and $\{v, y\}$ are the forcing subsets of $f_{G,D}(S_2)$. Therefore, $f_{G,D}(S_2) = 1$. Hence $f_{G,D}(G) = min\{1, 1\} = 1$. For G, we have, $0 < f_{G,D}(G) = 1 < \gamma_G(G) = 2$.

Remark 2.3 1. For every connected graph G, $0 \leqslant f_{G,D}(G) \leqslant \gamma_G(G)$.

2. Here the lower bound is sharp, since for any complete graph $S = V(G)$ is a unique γ_G-set. So, $T = \Phi$ is a forcing subset for S and $f_{G,D}(K_p) = 0$.

3. Example 2.2 proves the bounds are strict.

Theorem 2.4 *Let G be a connected graph. Then,*

(i) $f_{G,D}(G) = 0$ *if and only if G has a unique γ_G-set;*

(ii) $f_{G,D}(G) = 1$ *if and only if G has at least two γ_G-sets, one of which, say, S has forcing (G,D)-number equal to* 1;

(iii) $f_{G,D}(G) = \gamma_G(G)$ *if and only if every* γ_G*-set* S *of* G *has the property,* $f_{G,D}(S) = |S| = \gamma_G(G)$.

Proof (*i*) Suppose $f_{G,D}(G) = 0$. Then, by Definition 2.1, $f_{G,D}(S) = 0$ for some γ_G-set S of G. So, empty set is a minimum forcing subset for S. But, empty set is a subset of every set. Therefore, by Definition 2.1, S is the unique γ_G-set of G. Conversely, let S be the unique γ_G-set of G. Then, empty set is a minimum forcing subset of S. So, $f_{G,D}(G) = 0$.

(*ii*) Assume $f_{G,D}(G) = 1$. Then, by (*i*), G has at least two γ_G-sets. $f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is any } \gamma_{\rm G} - \text{setof } G\}$. So, $f_{G,D}(S) = 1$ for at least one γ_G-set S. Conversely, suppose G has at least two γ_G-sets satisfying the given condition. By (*i*), $f_{G,D}(G) \neq 0$. Further, $f_{G,D}(G) \geqslant 1$. Therefore, by assumption, $f_{G,D}(G) = 1$.

(*iii*) Let $f_{G,D}(G) = \gamma_G(G)$. Suppose S is a γ_G-set of G such that $f_{G,D}(S) < |S| = \gamma_G(\text{G})$. So, S has a forcing subset T such that $|T| < |S|$. Therefore, $f_{G,D}(G) = \min\{f_{G,D}(S) : S \text{ is a } \gamma_{\rm G} - \text{set of } G\} \leqslant |T| < |S| = \gamma_G(G)$. This is a contradiction. So, every γ_G-set S of G satisfies the given condition. The converse is obvious. Hence the result. □

Corollary 2.5 $f_{G,D}(P_n) = 0$ *if* $n \equiv 1(mod3)$.

Proof Let $P_n = (v_1, v_2, \ldots, v_{3k+1})$, $k \geqslant 0$. Now, $S = \{v_1, v_4, v_7, \ldots, v_{3k+1}\}$ is the unique γ_G-set of P_n. So, by Theorem 2.4, $f_{G,D}(P_n) = 0$. □

Observation 2.6 *Let* G *be any graph with at least two* γ_G*-sets. Suppose* G *has a* γ_G*-set* S *satisfying the following property:*

S has a vertex u such that $u \in S'$ for every γ_G-set S' different from S *(I),*

Then, $f_{G,D}(G) = 1$.

Proof As G has at least two γ_G-sets, by Theorem 2.4, $f_{G,D}(G) \neq 0$. If G satisfies (I), then we observe that $f_{G,D}(S) = 1$. So, by Definition 2.1, $f_{G,D}(G) = 1$. □

Corollary 2.7 *Let* G *be any graph with at least two* γ_G*-sets. Suppose* G *has a* γ_G*-set* S *such that* $S \bigcap S' = \phi$ *for every* γ_G*-set* S' *different from* S*. Then* $f_{G,D}(G) = 1$.

Proof Given that G has a γ_G-set S such that $S \bigcap S' = \phi$ for every γ_G-set S' different from S. Then, we observe that S satisfies property (I) in Observation 2.6. Hence, we have, $f_{G,D}(G) = 1$. □

Corollary 2.8 *Let* G *be any graph with at least two* γ_G*-sets. If pair wise intersection of distinct* γ_G*-sets of* G *is empty, then* $f_{G,D}(G) = 1$.

Proof The proof proceeds along the same lines as in Corollary 2.7. □

Corollary 2.9 $f_{G,D}(C_n) = 1$ *if* $n = 3k,\ k > 1$.

Proof Let $n = 3k,\ k > 1$. Let $V(C_n) = \{v_1, v_2, \ldots, v_{3k}\}$. Note that the only γ_G-sets of C_n are $S_1 = \{v_1, v_4, \ldots, v_{3(k-1)+1}\}$, $S_2 = \{v_2, v_5, \ldots, v_{3(k-1)+2}\}$ and $S_3 = \{v_3, v_6, \ldots, v_{3k}\}$.

Further, we have, $S_1 \bigcap S_2 = S_1 \bigcap S_3 = S_2 \bigcap S_3 = \emptyset$. That is, pair wise intersection of distinct γ_G-sets of C_n is empty. Hence, from Corollary 2.8, we have $f_{G,D}(C_n) = 1$ if $n = 3k$. □

Definition 2.10 *A vertex v of G is said to be a (G,D)-vertex of G if v belongs to every γ_G-set of G.*

Remark 2.11 1. All the extreme vertices of a graph G are (G,D)-vertices of G.

2. If G has a unique γ_G-set S, then every vertex of S is a (G,D)-vertex of G.

Lemma 2.12 *Let $G = (V, E)$ be any graph and $u \in V(G)$ be a (G,D)-vertex of G. Suppose S is a γ_G-set of G and T is a minimum forcing subset of S, then $u \notin T$.*

Proof Since u is a (G,D)-vertex of G, u is in every γ_G-set of G. Given that S is a γ_G-set of G and T is a minimum forcing subset of S. Suppose $u \in T$. Then, there exists a γ_G-set S' of G different from S such that $T - \{u\} \subseteq S'$. Otherwise, $T - \{u\}$ is a forcing subset of S. Since $u \in S'$, $T \subseteq S'$. This contradicts the fact that T is a minimum forcing subset of S. Hence, from the above arguments, we have $u \notin T$. □

Corollary 2.13 *Let W be the set of all (G,D)-vertices of G. Suppose S is a γ_G-set of G and T is a forcing subset of S. If W is non-empty, then $T \neq S$.*

Definition 2.14 *Let G be a connected graph and S be a γ_G-set of G. Suppose T is a minimum forcing subset of S. Let $E = S - T$ be the relative complement of T in its relative γ_G-set S. Then, $\mathscr{L}$ is defined by*

$$\begin{aligned} \mathscr{L} &= \{E | E \; is \; a \; relative \; complement \; of \; a \; minimum \\ &\quad forcing \; subset \; T \; in \; its \; relative \; \gamma_G - set \; S \; of \; G\}. \end{aligned}$$

Theorem 2.15 *Let G be a connected graph and ζ = The intersection of all $E \in \mathscr{L}$. Then, ζ is the set of all (G,D)-vertices of G.*

Proof Let W be the set of all (G,D)-vertices of G.

Claim $W = \zeta$, the intersection of all $E \in \mathscr{L}$. Let $v \in W$. By Definition 2.10, v is in every γ_G-set of G. Let S be a γ_G-set of G and T be a minimum forcing subset of S. Then, $v \in S$. From Lemma 2.12, we have, $v \notin T$. So, $v \in E = S - T$. Hence, $v \in E$ for every $E \in \mathscr{L}$. That is, $v \in \zeta$. Conversely, let $v \in \zeta$. Then, $v \in E = S - T$, where T is a minimum forcing subset of the γ_G-set S. So, $v \in S$ for every γ_G-set S of G. That is, $v \in W$. □

Corollary 2.16 *Let S be a γ_G-set of a graph G and T is a minimum forcing subset of S. Then, $W \bigcap T = \emptyset$.*

Remark 2.17 *The above result holds even if G has a unique γ_G-set.*

Corollary 2.18 *Let W be the set of all (G,D)-vertices of a graph G. Then, $f_{G,D}(G) \leqslant \gamma_G(G) - |W|$.*

Remark 2.19 In the above corollary, the inequality is strict. For example, consider the following graph G.

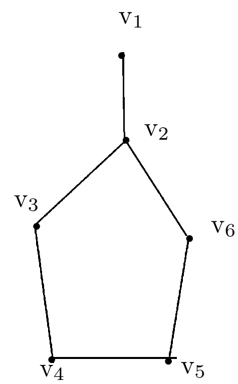

Fig.2.2

For G, $S_1 = \{v_1, v_4, v_5\}, S_2 = \{v_1, v_3, v_5\}, S_3 = \{v_1, v_4, v_6\}$ are the only distinct γ_G-sets. Therefore, $\gamma_G(G) = 3$. But, $f_{G,D}(S_1) = 2$ and $f_{G,D}(S_2) = f_{G,D}(S_3) = 1$. So, $f_{G,D}(G) = \min\{f_{G,D}(S)$: S is a γ_G-set of $G\} = 1$. Also, $W = \{1\}$. Now, $\gamma_G(G) - |W| = 3 - 1 = 2$. Hence $f_{G,D}(G) \leqslant \gamma_G(G) - |W|$.

Also the upper bound is sharp. For example, consider the following graph G.

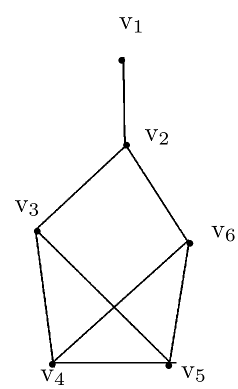

Fig.2.3

For G, $S_1 = \{v_1, v_4, v_5\}, S_2 = \{v_1, v_3, v_6\}$ are different γ_G-sets. Therefore, $\gamma_G(G) = 3$. But, $f_{G,D}(S_1) = f_{G,D}(S_2) = 2$. So, $f_{G,D}(G) = \min\{f_{G,D}(S)$: S is a γ_G-set of $G\} = 2$. Also, $W = \{1\}$. Now, $\gamma_G(G) - |W| = 3 - 1 = 2$. Hence, $f_{G,D}(G) = \gamma_G(G) - |W|$.

Corollary 2.20 *$f_{G,D}(G) \leqslant \gamma_G(G) - k$ where k is the number of extreme vertices of G.*

Proof The result follows from $|W| \geqslant k$. □

Theorem 2.21 *For a complete graph $G = K_p$, $f_{G,D}(G) = 0$ and $|W| = p$.*

Proof $V(K_p)$ is the unique γ_G-set of K_p. Hence by Theorem 2.4, $f_{G,D}(K_p) = 0$. By Remark 2.11, $W = V(G)$ with $|W| = p$. □

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Lucas Gracefulness of Almost and Nearly for Some Graphs

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Abstract: Let G be a (p,q) - graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}, (a \in N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits Lucas graceful labeling, then G is said to be Lucas graceful graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_{a-1}, l_{a+1}\}$, $(a \in N)$, is said to be almost Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\} or \{l_1, l_2, \cdots, l_{q-1}, l_{q+1}\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. Then G is called almost Lucas graceful graph if it admits almost Lucas graceful labeling. Also, an injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $(a \in N)$, is said to be nearly Lucas graceful labeling if the induced edge labeling $f_1(u,v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \cdots, l_{i-1}, l_{i+1}, l_{i+2}, \cdots, l_{j-1}, l_{j+1}, l_{j+2}, \cdots, l_{k-1}, l_{k+1}, l_{k+2}, \cdots, l_b$ $(b \in N$ and $b \leq a)$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be nearly Lucas graceful graph. In this paper, we show that the graphs $S_{m,n}, S_{m,n}@P_t$ and $F_m@P_n$ are almost Lucas graceful graphs. Also we show that the graphs $S_{m,n}@P_t$ and C_n are nearly Lucas graceful graphs.

Key Words: Smarandache-Fibonacci triple, super Smarandache-Fibonacci graceful graph, graceful labeling, Lucas graceful labeling, almost Lucas graceful labeling and nearly Lucas graceful labeling.

AMS(2010): 05C78

§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A cycle of length n is denoted by C_n $\cdot G^+$ is a graph obtained from the graph G by attaching pendant vertex to each vertex of G. The concept of graceful labeling was introduced by Rosa [3] in 1967. A function f is called a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{1, 2, 3, \cdots, q\}$ such that when each edge uv is assigned the label

[1]Received May 26, 2011. Accepted September 6, 2011.

$|f(u) - f(v)|$, the resulting edge labels are distinct. The notion of Fibonacci graceful labeling was introduced by K.M.Kathiresan and S.Amutha [4]. We call a function f, a Fibonacci graceful label labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \cdots, F_q\}$, where F_q is the q^{th} Fibonacci number of the Fibonacci series $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \cdots$ and each edge uv is assigned the label $|f(u) - f(v)|$. Based on the above concept we define the following.

A *Smarandache-Fibonacci triple* is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Clearly, it is a generalization of *Fibonacci sequence* and *Lucas sequence*. Let G be a (p, q)-graph and $\{S(n)|n \geq 0\}$ a Smarandache-Fibonacci triple. An bijection $f\colon V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. Particularly, if $S(n), n \geq 0$ is just the Lucas sequence, such a labeling $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$ $(a \in N)$ is said to be *Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection on to the set $\{l_1, l_2, \cdots, l_q\}$. If G admits Lucas graceful labeling, then G is said to be *Lucas graceful graph*. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_{a-1}, l_{a+1}\}$, $(a \in N)$, is said to be *almost Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u)-f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ or $\{l_1, l_2, \cdots, l_{q-1}, l_{q+1}\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. Then G is called *almost Lucas graceful graph* if it admits almost Lucas graceful labeling. Also, an injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $(a \in N)$, is said to be *nearly Lucas graceful labeling* if the induced edge labeling $f_1(u, v) = |f(u) - f(v)|$ onto the set $\{l_1, l_2, \cdots, l_{i-1}, l_{i+1}, l_{i+2}, \cdots, l_{j-1}, l_{j+1}, l_{j+2}, \cdots, l_{k-1}, l_{k+1}, l_{k+2}, \cdots, l_b$ $(b \in N$ and $b \leq a)$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be *nearly Lucas graceful graph*. In this paper, we show that the graphs $S_{m,n}, S_{m,n}@P_t$ and $F_m@P_n$ are almost Lucas graceful graphs. Also we show that the graphs $S_{m,n}@P_t$ and C_n are nearly Lucas graceful graphs.

§2. Almost Lucas Graceful Graphs

In this section, we show that some graphs namely $S_{m,n}, S_{m,n}@P_t$ and $F_m@P_n$ are almost Lucas graceful graphs.

Definition 2.1 *Let G be a (p, q) - graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_{a-1}, l_{a+1}\}$, $a \in N$, is said to be almost Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ or $\{l_1, l_2, \cdots, l_{q-1}, l_{q+1}\}$. Then G is called almost Lucas graceful graph if it admits almost Lucas graceful labeling.*

Definition 2.2 ([2]) *$S_{m,n}$ denotes a star with n spokes in which each spoke is a path of length m.*

Theorem 2.3 *$S_{m,n}$ is an almost Lucas graceful graph when $m \equiv 1(mod\ 2)$ and $n \equiv 0(mod\ 3)$*

Proof Let $G = S_{m,n}$. Let $V(G) = \{u_{i,j} : 1 \leq i \leq m$ and $1 \leq j \leq n\}$ be the vertex set of

G. Let $E(G) = \{u_0u_{i,1} : 1 \le i \le m\} \cup \{u_{i,j}u_{i,j+1} : 1 \le i \le m,\ 1 \le j \le n-1\}$ be the edge set of G. So, $|V(G)| = mn+1$ and $|E(G)| = mn$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}, a \,\epsilon\, N$ by $f(u_0) = l_0$. For $i = 1, 2, \cdots, m-2$ and $i \equiv 1(mod\ 2)$, $f(u_{i,j}) = l_{n(i-1)+2j-1}, 1 \le j \le n$. For $i = 1, 2, \cdots, m-1$ and $i \equiv 0(mod\ 2)$, $f(u_{i,j}) = l_{ni+2-2j},\ 1 \le j \le n$. For $s = 1, 2, \cdots, \dfrac{n-3}{3}$ $f(u_{m,j}) = l_{(m-1)n+2(j+1)-3s},\ 3s-2 \le j \le 3s$. and for $s = \dfrac{n}{3}$, $f(u_{m,j}) = l_{(m-1)n+2(j+1)-3s},\ 3s-2 \le j \le 3s-1$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|l_0 - l_{n(i-1)+1}\right|\right\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{n(i-1)+1}\right\} \\ &= \left\{l_1, l_{2n+1}, l_{4n+1}, \cdots, l_{(m-1)n+1}\right\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{|l_0 - l_{ni}|\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{l_{ni}\} \\ &= \left\{l_{2n}, l_{4n}, ..., l_{(m-1)n}\right\}, \end{aligned}$$

$$\begin{aligned} E_3 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1}|\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \left\{\left|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}\right|\right\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \left\{l_{n(i-1)+2j}\right\} \\ &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{n(i-1)+2}, l_{n(i-1)+4}, \cdots, l_{n(i-1)+2n-2}\right\} \\ &= \{l_2, l_4, \cdots, l_{2n-2}\} \bigcup \{l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}\} \bigcup \\ &\quad \cdots \bigcup \left\{l_{(m-3)n+2}, l_{(m-3)n+4}, \cdots, l_{(m-3)n+2n-2}\right\} \\ &= \left\{l_2, l_4, \cdots, l_{2n-2}, l_{2n+4}, \cdots, l_{4n-2}, \cdots, l_{(m-3)n+2}, l_{(m-3)n+4}, \cdots, l_{mn-n-2}\right\}, \end{aligned}$$

$$
\begin{aligned}
E_4 &= \bigcup_{\substack{i=1 \\ i\equiv 0 (mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{ij+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0 (mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0 (mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} = \bigcup_{\substack{i=1 \\ i\equiv 0 (mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 0 (mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-1}, l_{ni-3}, \cdots, l_{ni-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \cdots, l_3\} \bigcup \{l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}\} \bigcup \\
&\quad \cdots \bigcup \{l_{(m-1)n-1}, l_{(m-1)n-3}, \cdots, l_{mn-3n+3}\} \\
&= \left\{l_{2n-1}, l_{2n-3}, \cdots, l_3, l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}, \cdots, l_{(m-1)n-1}, l_{(m-1)n-3}, \cdots, l_{mn-3n+3}\right\},
\end{aligned}
$$

$$
\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \left\{\left|l_{n(m-1)+2j-3s+2} - l_{n(m-1)+2j-3s+4}\right| : 3s-2 \le j \le 3s-1\right\} \\
&= \bigcup_{s=1}^{\frac{n-3}{3}} \left\{l_{n(m-1)+2j-3s+3} : 3s-2 \le j \le 3s-1\right\} \\
&= \left\{l_{n(m-1)+2}, l_{n(m-1)+4}\right\} \bigcup \left\{l_{n(m-1)+5}, l_{n(m-1)+7}\right\} \bigcup \\
&\quad \cdots \bigcup \left\{l_{n(m-1)+2n-10-n+3+3}, l_{n(m-1)+2n-8-n+3+3}\right\} \\
&= \left\{l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \cdots, l_{n(m-1)+n-4}, l_{n(m-1)+n-2}\right\} \\
&= \left\{l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \cdots, l_{mn-4}, l_{mn-2}\right\}.
\end{aligned}
$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, \cdots, \dfrac{n-3}{3}$. Let

$$
\begin{aligned}
E_6 &= \bigcup_{s=1}^{\frac{n-3}{3}} \{f_1(u_{m,j}u_{m,j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n-3}{3}} \{|f(u_{m,j}) - f(u_{m,j+1}) : j = 3s|\} \\
&= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \cdots, |f(u_{m,n-3}) - f(u_{m,n-2})|\} \\
&= \left\{\left|l_{(m-1)n+8-3} - l_{(m-1)n+10-6}\right|, \left|l_{(m-1)n+14-6} - l_{(m-1)n+16-9}\right|, \right. \\
&\quad \left. \cdots, \left|l_{(m-1)n+2n-4-n+3} - l_{(m-1)n+2n-2-n}\right|\right\} \\
&= \left\{\left|l_{(m-1)n+5} - l_{(m-1)n+4}\right|, \left|l_{(m-1)n+8-l}l_{(m-1)}n+7\right|, \cdots, \left|l_{(m-1)n+n-1} - l_{(m-1)n+n-2}\right|\right\} \\
&= \{|l_{(m-1)n+3}, l_{(m-1)n+6}, \cdots, l_{(m-1)n+n-3}|\} \\
&= \left\{l_{(m-1)n+3}, l_{(m-1)n+6}, \cdots, l_{mn-3}\right\}.
\end{aligned}
$$

For $s = \dfrac{n}{3}$, let

$$\begin{aligned} E_7 &= \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \le j \le 3s-1\} = \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \le j \le 3s-1\} \\ &= \left\{\left|l_{(m-1)n+2n-2-n} - l_{(m-1)n+2n-n}\right|, \left|l_{(m-1)n+2n-n} - l_{(m-1)n+2n+2-n}\right|\right\} \\ &= \left\{\left|l_{(m-1)n+n-2} - l_{(m-1)n+n}\right|, \left|l_{(m-1)n+n} - l_{(m-1)n+n+2}\right|\right\} \\ &= \left\{l_{(m-1)n+n-1}, l_{(m-1)n+n+1}\right\} = \{l_{mn-1}, l_{mn+1}\}. \end{aligned}$$

Now, $E = \bigcup_{i=1}^{7} E_i = \{l_1, l_2, ..., l_{mn-1}, l_{mn+1}\}$. So, the edge labels of G are distinct. Therefore, f is an almost Lucas graceful labeling. Thus $G = S_{m,n}$ is an almost Lucas graceful graph, when $m \equiv 1(mod\ 2)$ and $n \equiv 0(mod\ 3)$. □

Example 2.4 An almost Lucas graceful labeling of $S_{7,9}$ is shown in Fig.2.1.

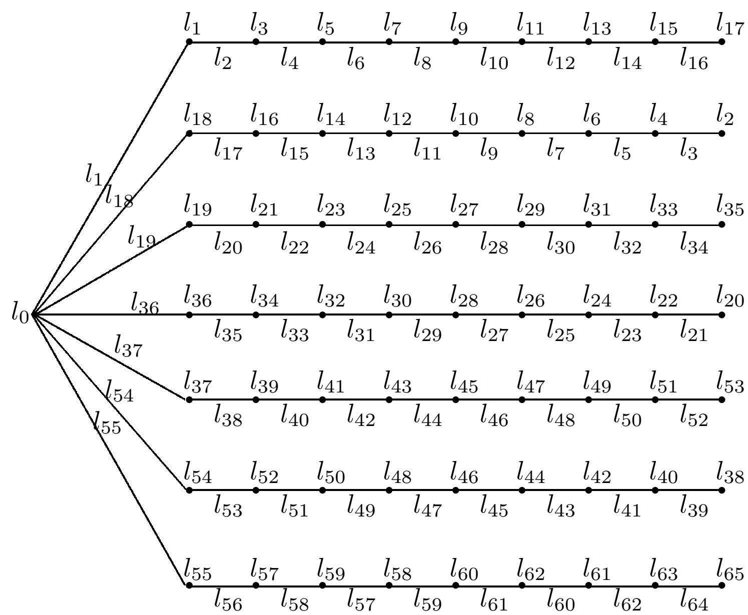

Fig.2.1 $S_{7,9}$

Definition 2.5([2]) *The graph $G = S_{m,n}@P_t$ consists of $S_{m,n}$ and a path P_t of length t which is attached with the maximum degree of the vertex of $S_{m,n}$.*

Theorem 2.6 *$S_{m,n}@P_t$ is an almost Lucas graceful graph when $m \equiv 0(mod\ 2)$ and $t \equiv 0(mod\ 3)$.*

Proof Let $G = S_{m,n}@P_t$ with $m \equiv 0(mod\ 3)$ and $t \equiv 0(mod\ 3)$. Let

$$V(G) = \{u_0, u_{i,j} : 1 \le i \le m \text{ and } 1 \le j \le n\} \bigcup \{v_k : 1 \le k \le t\},$$

$$\begin{aligned} E(G) &= \{u_0u_{i,1} : 1 \le i \le m\} \bigcup \{u_{i,j}u_{ij+1} : 1 \le i \le m \text{ and } 1 \le j \le n-1\} \\ &\bigcup \{u_0v_1\} \bigcup \{v_kv_{k+1} : 1 \le k \le t-1\} \end{aligned}$$

be the vertex set and edge set of G, respectively. Thus $|V(G)| = mn+t+1$ and $|E(G)| = mn+t$. Define $f : V(G) \rightarrow \{l_0, l_1, l_2, \cdots, l_a\}, a \in N$ by $f(u_0) = l_0$. For $i = 1, 2, \cdots, m$ and for $i \equiv 1(mod\ 2)$, $f(u_{i,j}) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$. For $i = 1, 2, \cdots, m$ and for $i \equiv 1(mod\ 2)$, $f(u_{i,j} = l_{ni-2j+2},\ 1 \leq j \leq n$. For $s = 1, 2, \cdots, \dfrac{t-3}{3}$, $f(v_k) = l_{mn+2k-3s+2},\ 3s-2 \leq k \leq 3s$ and for $s = \dfrac{t}{3}$, $f(v_k) = l_{mn+2k-3s+2},\ 3s-2 \leq k \leq 3s-1$. We claim that the edge labels are distinct. Let

$$\begin{aligned}
E_1 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|l_0 - l_{n(i-1)+1}\right|\right\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{n(i-1)+1}\right\} = \left\{l_1, l_{2n+1}, l_{4n+1}, \cdots, l_{n(m-1)+1}\right\},
\end{aligned}$$

$$\begin{aligned}
E_2 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \{|f(u_0) - f(u_{i,1})|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \{|l_0 - l_{ni}|\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \{l_{ni}\} = \{l_{2n}, l_{4n}, \cdots, l_{mn}\},
\end{aligned}$$

$$\begin{aligned}
E_3 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \left\{\left|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}\right|\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \left\{l_{n(i-1)+2j}\right\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{n(i-1)+2}, l_{n(i-1)+4}, \cdots, l_{n(i-1)+2n-2}\right\} \\
&= \{l_2, l_4, \cdots, l_{2n-2}\} \bigcup \{l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}\} \bigcup \\
&\quad \cdots \bigcup \left\{l_{n(m-2)+2}, l_{n(m-2)+4}, \cdots, l_{mn-2}\right\} \\
&= \{l_2, l_4, \cdots, l_{2n-2}, l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}, \cdots, l_{n(m-2)+2}, l_{n(m-2)+4}, \cdots, l_{mn-2}\},
\end{aligned}$$

$$
\begin{aligned}
E_4 &= \bigcup_{\substack{i=1 \\ i\equiv 1 (mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1 (mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{|f(u_{i,j}) - f(u_{i,j+1})|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1 (mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1 (mod\ 2)}}^{m} \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\} = \bigcup_{\substack{i=1 \\ i\equiv 0 (mod\ 2)}}^{m} \{l_{ni-1}, l_{ni-3}, \cdots, l_{ni-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \cdots, l_3\} \cup \{l_{4n-1}, l_{4n-3}, ..., l_{2n+3}\} \cup ... \cup \{l_{mn-1}, l_{mn-3}, \cdots, l_{mn-2n+3}\} \\
&= \{l_{2n-1}, l_{2n-3}, \cdots, l_3, l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}, \cdots, l_{mn-1}, l_{mn-3}, \cdots, l_{mn-2n+3}\},
\end{aligned}
$$

$$
E_1^{'} = \{f_1(u_0v_1)\} = \{|f(u_0) - f(v_1)|\} = \{|l_0 - l_{mn+1}|\} = \{l_{mn+1}\},
$$

$$
\begin{aligned}
E_2^{'} &= \bigcup_{s=1}^{\frac{t-3}{3}} \{f_1(v_kv_{k+1}) : 3s-2 \le k \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{|f(v_k) - f(v_{k+1}) : 3s-2 \le k \le 3s-1|\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{|l_{mn+2k+2-3s} - l_{mn+2k+4-3s}| : 3s-2 \le k \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{t-3}{3}} \{l_{mn+2k+3-3s} : 3s-2 \le k \le 3s-1\} \\
&= \{l_{mn+2}, l_{mn+4}\} \cup \{l_{mn+5}, l_{mn+7}\} \cup \cdots \cup \{l_{mn+t-4}, l_{mn+t-2}\} \\
&= \{l_{mn+2}, l_{mn+4}, l_{mn+5}, l_{mn+7}, \cdots, l_{mn+t-4}, l_{mn+t-2}\}
\end{aligned}
$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for integers $s = 1, 2, \cdots, \frac{t-3}{3}$. Let

$$
\begin{aligned}
E_3^{'} &= \bigcup_{s=1}^{\frac{t-3}{3}} \{f_1(u_{3s}u_{3s+1})\} \\
&= \{|f(u_{3s}) - f(u_{3s+1})|\} \\
&= \{|f(u_3) - f(u_4)|, |f(u_6) - f(u_7)|, ..., |f_{(t-3)} - f_{(t-2)}|\} \\
&= \{|l_{mn+8-3} - l_{mn+10-6}|, |l_{mn+14-6} - l_{mn+16-9}|, ..., |l_{mn+2t-4-t+3} - l_{mn+2t-2-t}|\} \\
&= \{|l_{mn+5} - l_{mn+4}|, |l_{mn+8} - l_{mn+7}|, ..., |l_{mn+t-1} - l_{mn+t-2}|\} \\
&= \{l_{mn+3}, l_{mn+6}, ..., l_{mn+t-3}\}.
\end{aligned}
$$

For $s = \dfrac{t}{3}$, let

$$\begin{aligned} E_4^{'} &= \{f_1(v_k v_{k+1}) : 3s-2 \le k \le 3s-1\} \\ &= \{|f(v_k) - f(v_{k+1})| : 3s-2 \le k \le 3s-1\} \\ &= \{|l_{mn+2t-4+2-t} - l_{mn+2t-2+2-t}|, |l_{mn+2t-2+2-t} - l_{mn+2t+2-t}|\} \\ &= \{|l_{mn+t-2} - l_{mn+t+2}| \, |l_{mn+t} - l_{mn+t+1}|\} = \{l_{mn+t-1}, l_{mn+t+1}\}. \end{aligned}$$

Now, $E = \bigcup\limits_{i=1}^{4} (E_i \bigcup E_i^{'}) = \{l_1, l_2, \cdots, l_{mn}, \cdots, l_{mn+t-1}, l_{mn+t+1}\}$. So, the edge labels of G are distinct. Therefore, f is an almost Lucas graceful graph. Thus $G = S_{m,n}@P_t$ is an almost Lucas graceful graph when $m \equiv 0(mod\ 2)$ and $t \equiv 0(mod\ 3)$.

Example 2.7 An almost Lucas graceful labeling on $S_{4,7}@P_6$ is shown in Fig.2.2.

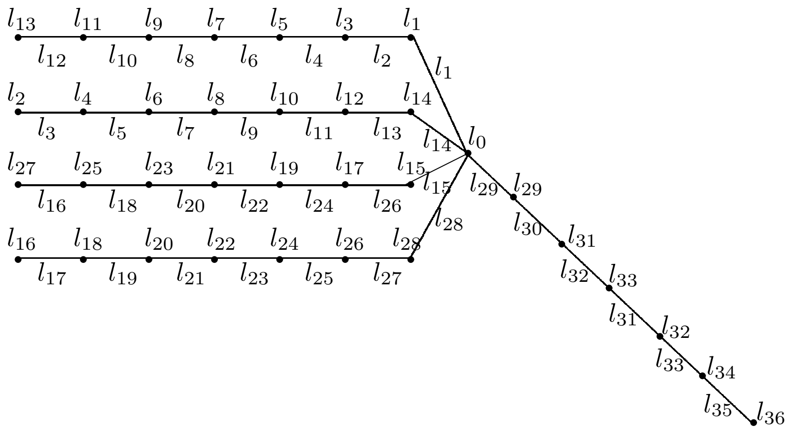

Fig.2.2 $S_{4,7}@P_6$

Definition 2.8([2]) *The graph $G = F_m@P_n$ consists of a fan F_m and a path P_n of length n which is attached with the maximum degree of the vertex of F_m.*

Theorem 2.9 *$F_m@P_n$ is almost Lucas graceful graph when $n \equiv 0(mod\ 3)$.*

Proof Let $v_1, v_2, \cdots, v_{m+1}$ and u_0 be the vertices of a Fan F_m. Let $u_1, u_2, \cdots, u_n$ be the vertices of a path P_n. Let $G = F_m@P_n$, $|V(G)| = m + n + 2$ and $|E(G)| = 2m + n + 1$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_{q+2}$ by $f(u_0) = l_0$; $f(v_i) = l_{2i-1}$; $f(u_j) = l_{2m+2j-3s+3}$, $3s-2 \le j \le 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{i=1}^{m} \{f_1(v_i v_{i+1})\} = \bigcup_{i=1}^{m} \{|f(v_i) - f(v_{i+1})|\} \\ &= \bigcup_{i=1}^{m} \{|l_{2i-1} - l_{2i+1}|\} \\ &= \bigcup_{i=1}^{m} \{l_{2i}\} = \{l_2, l_4, ..., l_{2m}\}, \end{aligned}$$

$$\begin{aligned}E_2 &= \bigcup_{i=1}^{m+1}\{f_1(u_0v_i)\} = \bigcup_{i=1}^{m+1}\{|f(u_0)-f(v_i)|\}\\ &= \bigcup_{i=1}^{m+1}\{|l_0-l_{2i-1}|\} = \bigcup_{i=1}^{m+1}\{l_{2i-1}\} = \{l_1,l_3,...,l_{2m+1}\},\end{aligned}$$

$$E_3 = \{f_1(u_0u_1)\} = \{|f(u_0)-f(u_1)|\} = \{|l_0-l_{2m+2}|\} = \{l_{2m+2}\},$$

$$\begin{aligned}E_4 &= \bigcup_{s=1}^{\frac{n-3}{3}}\{f_1(u_ju_{j+1}) : 3s-2\le j\le 3s-1\}\\ &= \bigcup_{s=1}^{\frac{n-3}{3}}\{|f(u_j)-f(u_{j+1})| : 3s-2\le j\le 3s-1\}\\ &= \{|f(u_1)-f(u_2)|,|f(u_2)-f(u_3)|\}\bigcup\{|f(u_4)-f(u_5)|,|f(u_5)-f(u_6)|\}\bigcup\\ &\quad\cdots\bigcup\{|f(u_{n-5})-f(u_{n-4})|,|f(u_{n-4})-f(u_{n-3})|\}\\ &= \{|l_{2m+2}-l_{2m+4}|,|l_{2m+4}-l_{2m+6}|\}\bigcup\{|l_{2m+5}-l_{2m+7}|,|l_{2m+7}-l_{2m+9}|\}\bigcup\\ &\quad\cdots\bigcup\{|l_{2m+2n-10+3-n+3}-l_{2m+2n-8+3-n+3}|,|l_{2m+2n-8+3-n+3}l_{2m+2n-6+3-n+3}|\}\\ &= \{l_{2m+3},l_{2m+5}\}\cup\{l_{2m+6},l_{2m+8}\}\cup\cdots\cup\{l_{2m+n-3},l_{2m+n-1}\}\\ &= \{l_{2m+3},l_{2m+5},l_{2m+6},l_{2m+8},\cdots,l_{2m+n-3},l_{2m+n-1}\}.\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for $s=1,2,\cdots,\frac{n}{3}-1$. Let

$$\begin{aligned}E_5 &= \bigcup_{s=1}^{\frac{n}{3}-1}\{f_1(u_ju_{j+1}) : j=3s\} = \bigcup_{s=1}^{\frac{n}{3}-1}\{|f(u_j)-f(u_{j+1})| : j=3s\}\\ &= \{|l_{2m+6+3-3}-l_{2m+8+3-6}|,|l_{2m+12+3-6}-l_{2m+14+3-9}|,\\ &\quad\cdots,|l_{2m+2n-6+3-n+3}-l_{2m+2n-4+3-n}|\}\\ &= \{|l_{2m+6}-l_{2m+5}|,|l_{2m+9}-l_{2m+8}|,|l_{2m+n}-l_{2m+n-1}|\}\\ &= \{l_{2m+4},l_{2m+7},\cdots,l_{2m+n-2}\}.\end{aligned}$$

For $s=\frac{n}{3}$, let

$$\begin{aligned}E_6 &= \{f_1(u_ju_{j+1}) : 3s-2\le j\le 3s-1\}\\ &= \{|f(u_j)-f(u_{j+1})| : 3s-2\le j\le 3s-1\}\\ &= \{|f(u_{n-2})-f(u_{n-1})|,|f(u_{n-1})-f(u_n)|\}\\ &= \{|l_{2m+2n-4+3-n}-l_{2m+2n-2+3-n}|,|l_{2m+2n-2+3-n}-l_{2m+2n+3-n}|\}\\ &= \{|l_{2m+n-1}-l_{2m+n+1}|,|l_{2m+n+1}-l_{2m+n+3}|\}\\ &= \{l_{2m+n},l_{2m+n+2}\}.\end{aligned}$$

Now, $E=\bigcup_{i=1}^{6}E_i = \{l_1,l_2,\cdots,l_{2m},l_{2m+1},l_{2m+2},\cdots,l_{2m+n-2},l_{2m+n-1},l_{2m+n},l_{2m+n+2}\}$. So, the edge labels of G are distinct. Therefore, f is an almost Lucas graceful labeling.
Thus $G=F_m@P_n$ is an almost Lucas graceful graph when $n\equiv 0(mod\ 3)$. □

Example 2.10 An almost Lucas graceful labeling on $F_5@P_6$ is shown in Fig.2.3.

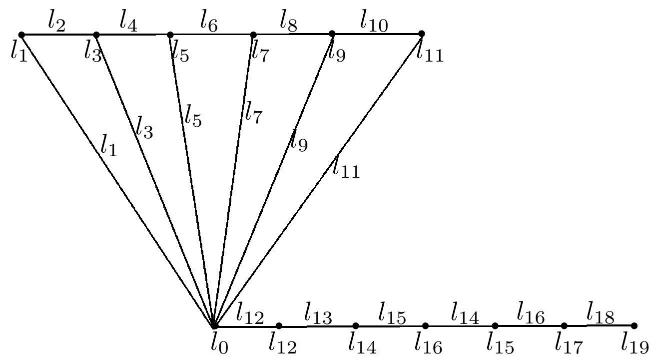

Fig.2.3 $F_5@P_6$

§3. Nearly Lucas Graceful Graphs

In this section, we show that the graphs $S_{m,n}@P_t$ and C_n are nearly Lucas graceful graphs.

Definition 3.1 *Let G be a (p,q) - graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $(a \in N)$, is said to be nearly Lucas graceful labeling if the induced edge labeling $f_1(u,v) = |f(u)-f(v)|$ onto the set $\{l_1, l_2, \cdots, l_{i-1}, l_{i+1}, l_{i+2}, \cdots, l_{j-1}, l_{j+1}, l_{j+2}, \cdots, l_{k-1}, l_{k+1}, l_{k+2}, \cdots, l_b$ $(b \in N$ and $b \leq a)$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11$, etc.. If G admits nearly Lucas graceful labeling, then G is said to be nearly Lucas graceful graph.*

Theorem 3.2 *$S_{m,n}@P_t$ is a nearly Lucas graceful graph when $n \equiv 1, 2(mod\ 3)$ $m \equiv 1(mod\ 2)$ and $t = 1, 2(mod\ 3)$*

Proof Let $G = S_{m,n}@P_t$ with $V(G) = \{u_0, u_{i,j} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \cup \{v_k : 1 \leq k \leq t\}$. Let $E(G) = \{u_0u_{i,j} : 1 \leq i \leq m\} \bigcup \{u_{i,j}u_{i,j+1} : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \bigcup \{u_0v_1\} \cup \{v_kv_{k+1} : 1 \leq k \leq t-1\}$ be the edge set of G. So, $|V(G)| = mn + t + 1$ and $|E(G)| = mn + t$. Define $f : V(G) \to \{l_0, l_1, \cdots, l_a\}$, $a \in N$ by $f(u_0) = l_0$. For $i = 1, 2, \cdots, m$ and for $i \equiv 1(mod\ 2)$ $f(u_{i,j}) = l_{n(i-1)+2j-1}$, $1 \leq j \leq n$. For $i = 1, 2, \cdots, m$ and for $i \equiv 0(mod\ 2)$, $f(u_{i,j}) = l_{in-2j+2}$, $1 \leq j \leq n$. For $s = 1, 2, \cdots, \dfrac{n-2}{3} - 1$ or $s = 1, 2, \cdots, \dfrac{n-1}{3} - 1$ or $s = 1, 2, 3, \cdots, \dfrac{n}{3} - 1$, $f(u_{m,j}) = l_{mn+2(j+1)-3s}$, $3s-2 \leq j \leq 3s$. For $s = \dfrac{n-2}{3}$ or $\dfrac{n-1}{3}$ or $\dfrac{n}{3}$, $f(u_{m,j} = l_{mn+2(j+1)-3s}$, $3s-2 \leq j \leq 3s-1$. For $r = 1, 2, \cdots, \dfrac{t-2}{3}$ or $r = 1, 2, \cdots, \dfrac{t-1}{3}$ or $r = 1, 2, 3, \cdots, \dfrac{t}{3}$, $f(v_k) = l_{mn+2k+3-3r}$, $3r-2 \leq j \leq 3r-1$. We claim that the edge labels

are distinct. Let

$$\begin{aligned}
E_1 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{f_1(u_0u_{i,1}\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \{|f(u_0)-f(u_{i,j})|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|l_0 - l_{(i-1)n+1}\right|\right\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{(i-1)n+1}\right\} = \left\{l_1, l_{2n+1}, \cdots, l_{(m-1)n+1}\right\},
\end{aligned}$$

$$\begin{aligned}
E_2 &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{f_1(u_0u_{i,1})\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{|f(u_0)-f(u_{i,1})|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{l_0 - l_{in}\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m} \{l_{in}\} = \left\{l_{2n}, l_{4n}, \cdots, l_{(m-1)n}\right\},
\end{aligned}$$

$$\begin{aligned}
E_3 &= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \{|f(u_{i,j})-f(u_{i,j+1})|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \left\{\left|l_{(i-1)n+2j-1} - l_{(i-1)n+2j+1}\right|\right\} = \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \bigcup_{j=1}^{n-1} \left\{l_{(i-1)n+2j}\right\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{(i-1)n+2}, l_{(i-1)n+4}, \cdots, l_{(i-1)n+2n-2}\right\}\\
&= \{l_2, l_4, \cdots, l_{2n-2}\} \bigcup \{l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}\} \bigcup\\
&\quad \cdots \bigcup \left\{l_{(m-3)n+2}, l_{(m-3)n+4}, \cdots, l_{mn-n-2}\right\}\\
&= \left\{l_2, l_4, \cdots, l_{2n-2}, l_{2n+2}, l_{2n+4}, \cdots, l_{4n-2}, \cdots, l_{(m-3)+2}, l_{(m-3)n+4}, \cdots, l_{mn-n-2}\right\},
\end{aligned}$$

$$\begin{aligned}
E_4 &= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{f_1(u_{i,j}u_{i,j+1})\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{|f(u_{i,j})-f(u_{i,j+1})|\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{|l_{ni-2j+2} - l_{ni-2j}|\} = \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \bigcup_{j=1}^{n-1} \{l_{ni-2j+1}\}\\
&= \bigcup_{\substack{i=1\\ i\equiv 0(mod\ 2)}}^{m-1} \{l_{in-1}, l_{in-3}, \cdots, l_{in-2n+3}\}\\
&= \{l_{2n-1}, l_{2n-3}, \cdots, l_3\} \bigcup \{l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}\} \bigcup\\
&\quad \cdots \bigcup \left\{l_{(m-1)n-1}, l_{(m-1)n-3}, \cdots, l_{mn-3n+3}\right\}\\
&= \left\{l_{2n-1}, l_{2n-3}, \cdots, l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}, \cdots, l_{(m-1)n-1}, l_{(m-1)n-3}, \cdots, l_{mn-3n+3}\right\}.
\end{aligned}$$

For $n \equiv 1(mod\ 3)$ and $s = 1, 2, \cdots, \dfrac{n-4}{3}$, let

$$\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-4}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-4}{3}} \left\{\left|l_{(m-1)n+2j-3s+2} - l_{(m-1)n+2j-3s+4}\right| : 3s-2 \leq j \leq 3s-1\right\}
\end{aligned}$$

$$\begin{aligned}
&= \bigcup_{s=1}^{\frac{n-4}{3}} \left\{l_{(m-1)n+2j-3s+3} : 3s-2 \leq j \leq 3s-1\right\} \\
&= \left\{l_{(m-1)n+2}, l_{(m-1)n+4}\right\} \cup \left\{l_{(m-1)n+5}, l_{(m-1)n+7}\right\} \cup ... \cup \left\{l_{(m-1)n+n-4}, l_{(m-1)n+n-2}\right\} \\
&= \left\{l_{(m-1)n+2}, l_{(m-1)n+4}, l_{(m-1)n+5}, l_{(m-1)n+7}, ..., l_{mn-4}, l_{mn-2}\right\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop for integers $s = 1, 2, \cdots, \dfrac{n-4}{3}$. Let

$$\begin{aligned}
E_6 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_{m,j}u_{m,j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : j = 3s\} \\
&= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \cdots, |f(u_{m,n-1}) - f(u_{m,n})|\} \\
&= \left\{\left|l_{(m-1)n+5} - l_{(m-1)n+4}\right|, \left|l_{(m-1)n+7}\right|\ , \cdots, \left|l_{(m-1)n+2n-2-n+1} - l_{(m-1)n+2n+2-n-2}\right|\right\} \\
&= \left\{l_{(m-1)n+3}, l_{(m-1)n+6}, \cdots, l_{mn-1}\right\}.
\end{aligned}$$

For $s = \dfrac{n-1}{3}$, Let

$$\begin{aligned}
E_7 &= \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \left\{\left|l_{(m-1)n+2n-6+2-n+1} - l_{(m-1)n+2n-4+2-n+1}\right|,\right. \\
&\qquad \left.\left|l_{(m-1)n+2n-4+2-n+1} - l_{(m-1)n+2n-2+2-n+1}\right|\right\} \\
&= \{|l_{mn-3} - l_{mn-1}|, |l_{mn-1} - l_{mn+1}|\} = \{l_{mn-2}, l_{mn}\}
\end{aligned}$$

Now, $E = \bigcup_{i=1}^{7} E_i = \{l_1, l_2, \cdots, l_{mn}\}$. For $n \equiv 2(mod\ 3)$ and integers $s = 1, 2, \cdots, \dfrac{n-2}{3}$,

$$\begin{aligned}
E_1^{'} &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_{m,j}u_{m,j+1}) : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : 3s-2 \leq j \leq 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \left\{\left|l_{(m-1)n+2j+2-3s} - l_{(m-1)n+2j+4-3s}\right| : 3s-2 \leq j \leq 3s-1\right\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \left\{l_{(m-1)n+2j+3-3s} : 3s-2 \leq j \leq 3s-1\right\} \\
&= \left\{l_{(m-1)n+2}, l_{(m-1)n+4}\right\} \cup \left\{l_{(m-1)n+5}, l_{(m-1)n+7}\right\} \cup ... \cup \left\{l_{(m-1)n+n-3}, l_{(m-1)n+n-1}\right\} \\
&= \left\{l_{(m-1)n+2}, l_{(m-1)n+4}, l_{(m-1)n+5}, l_{(m-1)n+7}, \cdots, l_{mn-3}, l_{mn-1}\right\}
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $s+1^{th}$ loop for integers $s = 1, 2, \cdots, \dfrac{n-2}{3}$. Let

$$\begin{aligned}
E_2^{'} &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_{m,j}u_{m,j+1}) j = 3s\} = \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_{m,j}) - f(u_{m,j+1})| : j = 3s\} \\
&= \{|f(u_{m,3}) - f(u_{m,4})|, |f(u_{m,6}) - f(u_{m,7})|, \cdots, |f(u_{m,n-2}) - f(u_{m,n-1}|\} \\
&= \left\{\left|l_{(m-1)n+8-3} - l_{(m-1)n+10-6}\right|, \left|l_{(m-1)n+14-6} - l_{(m-1)n+16-9}\right|,\right. \\
&\quad \left.\cdots, \left|l_{(m-1)n+2n-2-n+2} - l_{(m-1)n+2n-n-1}\right|\right\} \\
&= \left\{\left|l_{(m-1)n+5} - l_{(m-1)n+4}\right|, \left|l_{(m-1)n+8} - l_{(m-1)n+7}\right|,\right. \\
&\quad \left.\cdots, \left|l_{(m-1)n+n} - l_{(m-1)n+n-1}\right|\right\} \\
&= \left\{l_{(m-1)n+3}, l_{(m-1)n+6}, \cdots, l_{mn-2}\right\}.
\end{aligned}$$

For $s = \dfrac{n+1}{3}$, let

$$\begin{aligned}
E_3^{'} &= \{f_1(u_{m,j}u_{m,j+1}) : j = 3s-2\} = \{|f(u_{m,j}) - f(u_{m,j+1})| : j = n-1\} \\
&= \{|f(u_{m,n-1}) - f(u_{m,n})|\} = \left\{\left|l_{(m-1)n+2n-n-1} - l_{(m-1)n+2n+2-n-1}\right|\right\} \\
&= \{|l_{mn-1} - l_{mn+1}|\} = \{l_{mn}\}.
\end{aligned}$$

Therefore, $E^{'} = \bigcup_{i=1}^{3} E_i^{'}$. Let

$$E_0 = \{f_1(u_0v_1)\} = \{|f(u_0) - f(v_1)|\} = \{|l_0 - l_{mn+2}|\} = \{l_{mn+2}\}.$$

For $t \equiv 2(mod\ 3)$ and $r = 1, 2, \cdots, \dfrac{t-2}{3}$, let

$$\begin{aligned}
E_1^{"} &= \bigcup_{r=1}^{\frac{t-2}{3}} \{f_1(v_k v_{k+1}) : 3r-2 \le k \le 3r-1\} \\
&= \bigcup_{r=1}^{\frac{t-2}{3}} \{|f(v_k) - f(v_{k+1})| : 3r-2 \le k \le 3r-1\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \bigcup \{|f(v_4) - f(v_5)|, |f(v_5) - f(v_6)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(v_{t-4}) - f(v_{t-3})|, |f(v_{t-3}) - f(v_{t-2})|\} \\
&= \{|l_{mn+3+2-3} - l_{mn+3+4-3}|, |l_{mn+3+4-3} - l_{mn+3+6-3}|\} \bigcup \\
&\quad \{|l_{mn+8+3-6} - l_{mn+10+3-6}|, |l_{mn+10+3-6} - l_{mn+12+3-6}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{mn+3+2t-8-t+2} - l_{mn+3+2t-6-t+2}|, |l_{mn+3+2t-6-t+2} - l_{mn+3+2t-4-t+2}|\} \\
&= \{|l_{mn+2} - l_{mn+4}|, |l_{mn+4} - l_{mn+6}|\} \bigcup \{|l_{mn+5} - l_{mn+7}|, |l_{mn+7} = l_{mn+9}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{mn+t-3} - l_{mn+t-1}|, |l_{mn+t-1} - l_{mn+t+1}|\} \\
&= \{l_{mn+3}, l_{mn+5}\} \bigcup \{l_{mn+6}, l_{mn+8}\} \bigcup \cdots \bigcup \{l_{mn+t-2}, l_{mn+t}\} \\
&= \{l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-2}, l_{mn+t}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop for integers $r = 1, 2, \cdots, \dfrac{t-2}{3}$. Let

$$\begin{aligned}
E_2^{"} &= \bigcup_{r=1}^{\frac{t-2}{3}} \{f_1(v_k v_{k+1}) : k = 3r\} = \bigcup_{r=1}^{\frac{t-2}{3}} \{|f(v_k) - f(v_{k+1})| : k = 3r\} \\
&= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, \cdots, |f(v_{t-2}) - f(v_{t-1})|\} \\
&= \{|l_{mn+3+6-3} - l_{mn+3+8-6}|, |l_{mn+3+12-6} - l_{mn+3+14-9}|, \\
&\quad \cdots, |l_{mn+3+2t-4-t+2} - l_{mn+3+2t-2-t-1}|\} \\
&= \{|l_{mn+6} - l_{mn+5}|, |l_{mn+9} - l_{mn+8}|, ..., |l_{mn+t+1} - l_{mn+t}|\} \\
&= \{l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t-1}\}.
\end{aligned}$$

For $s = \dfrac{t+1}{3}$, let

$$\begin{aligned}
E_3^{"} &= \{f_1(v_k v_{k+1}) : k = 3r-2\} = \{|f(v_k) - f(v_{k+1})| : k = 3r-2\} \\
&= \{|l_{mn+3+2t-2-t-1} - l_{mn+3+2t-t-1}|\} = \{|l_{mn+t} - l_{mn+t+2}|\} = \{l_{mn+t+1}\}
\end{aligned}$$

Therefore, $E" = E_0 \bigcup E_1^{"} \bigcup E_2^{"} \bigcup E_3^{"} = \{l_{mn+2}, l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-2}, l_{mn+t}, l_{mn+t+1}, l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t-1}\}$. Now, $E \bigcup E^{"} = \bigcup_{i=1}^{7} E_i \cup E_0 \cup E_1^{"} \cup E_2^{"} \cup E_3^{"} = \{l_1, l_2, \cdots, l_{mn}, l_{mn+2}, l_{mn+3}, l_{mn+4}, \cdots, l_{mn+t-2}, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. So, the edge labels of G are

distinct. For $t \equiv 1(mod\ 3)$ and integers $r = 1, 2, \cdots, \dfrac{t-1}{3}$, let

$$
\begin{aligned}
E_1^{'''} &= \bigcup_{r=1}^{\frac{t-1}{3}} \{f_1(v_k v_{k+1}) : 3r-2 \le k \le 3r-1\} \\
&= \bigcup_{r=1}^{\frac{t-1}{3}} \{|f(v_k) - f(v_{k+1})| : 3r-2 \le k \le 3r-1\} \\
&= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|\} \bigcup \{|f(v_4) - f(v_5)|, |f(v_5) - f(v_6)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(v_{t-3}) - f(v_{t-2})|, |f(v_{t-2}) - f(v_{t-1})|\} \\
&= \{|l_{mn+3+2-3} - l_{mn+3+4-3}|, |l_{mn+3+4-3} - l_{mn+3+6-3}|\} \\
&\quad \bigcup \{|l_{mn+3+8-6} - l_{mn+3+10-6}|, |l_{mn+3+10-6} - l_{mn+3+12-6}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{mn+3+2t-6-t+1} - l_{mn+3+2t-4-t+1}|, |l_{mn+3+2t-4-t+1} - l_{mn+3+2t-2-t+1}|\} \\
&= \{|l_{mn+2} - l_{mn+4}|, |l_{mn+4} - l_{mn+6}|\} \bigcup \{|l_{mn+5} - l_{mn+7}|, |l_{mn+7} - l_{mn+9}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{mn+t-2} - l_{mn+t}|, |l_{mn+t} - l_{mn+t+2}|\} \\
&= \{l_{mn+3}, l_{mn+5}, l_{mn+6}, l_{mn+8}, \cdots, l_{mn+t-1}, l_{mn+t+1}\}.
\end{aligned}
$$

We find the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop for integers $r = 1, 2, \cdots, \dfrac{t-1}{3}$. Let

$$
\begin{aligned}
E_2^{'''} &= \bigcup_{r=1}^{\frac{t-1}{3}} \{f_1(v_k v_{k+1}) : k = 3r\} \\
&= \bigcup_{r=1}^{\frac{t-1}{3}} \{|f(v_k) - f(v_{k+1})| : k = 3r\} \\
&= \{|f(v_3) - f(v_4)|, |f(v_6) - f(v_7)|, \cdots, |f(v_{t-1}) - f(v_t)|\} \\
&= \{|l_{mn+3+6-3} - l_{mn+3+8-6}|, \cdots, |l_{mn+3+2t-2-t+1} - l_{mn+3+2t-t-2}|\} \\
&= \{|l_{mn+6} - l_{mn+5}|, |l_{mn+9} - l_{mn+8}|, ..., |l_{mn+t+2} - l_{mn+t+1}|\} \\
&= \{l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t}\}
\end{aligned}
$$

Therefore $E^{'''} = E_0 \bigcup E_1^{'''} \bigcup E_2^{'''} = \{l_{mn+2}, l_{mn+3}, \cdots, l_{mn+t-1}, l_{mn+t+1}, l_{mn+4}, l_{mn+7}, \cdots, l_{mn+t}\} = \{l_{mn+2}, l_{mn+3}, l_{mn+4}, \cdots, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. Now, $E \cup E^{'} \cup E^{'''} = \bigcup\limits_{i=1}^{4} E_i \bigcup \left\{\bigcup\limits_{i=1}^{3} E_i^{'}\right\} \bigcup \left\{E_0 \bigcup E_1^{'''} \bigcup E_2^{'''}\right\} = \{l_1, l_2, \cdots, l_{mn}, l_{mn+2}, l_{mn+3}, \cdots, l_{mn+t-1}, l_{mn+t}, l_{mn+t+1}\}$. So, the edge labels of G are distinct. In both cases, f is a nearly Lucas graceful labeling. Thus $G = S_{m,n}@P_t$ is a nearly Lucas graceful graph when $m \equiv 1(mod\ 2)$, $n \equiv 1, 2(mod\ 3)$ and $t \equiv 1, 2, (mod\ 3)$.

Example 3.3 A nearly Lucas graceful labeling of $S_{5,7}@P_7$ is shown in Fig.3.1.

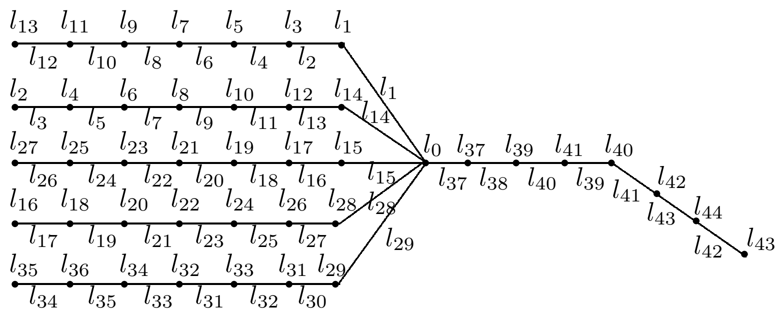

Fig.3.1 $S_{5,7}@P_7$

Theorem 3.4 *C_n is a nearly Lucas graceful graph. when $n \equiv 1, 2(mod\ 3)$.*

Proof Let $G = C_n$ with $V(G) = \{u_i : 1 \leq i \leq n\}$. Let $E(G) = \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ be the edge set of G. So, $|V(G)| = n$ and $|E(G)| = n$.

Case 1 $n \equiv 1(mod\ 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\},\ a\ \epsilon\ N$ by $f(u_1) = l_0$. For $s = 1, 2, \cdots, \dfrac{n-4}{3}$, $f(u_i) = l_{2i-3s},\ 3s-1 \leq i \leq 3s+1$ and for $s = \dfrac{n-1}{3}$, $f(u_i) = l_{2i-3s},\ 3s-1 \leq i \leq 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \{f_1(u_1u_2), f_1(u_nu_1)\} = \{|f(u_1) - f(u_2)|, |f(u_n - f(u_1)|\} \\ &= \{|l_0 - l_1|, |l_{2n-n+1} - l_0|\} = \{l_1, l_{n+1}\}. \end{aligned}$$

For $s = 1, 2, \cdots, \dfrac{n-1}{3}$, let

$$\begin{aligned} E_2 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_iu_{i+1})\ 3s-1 \leq i \leq 3s\} \\ &= \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\} \\ &= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \bigcup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \bigcup \\ &\quad \cdots \bigcup \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|\} \bigcup \{|l_4 - l_6|, |l_6 - l_8|\} \bigcup \\ &\quad \cdots \bigcup \{|l_{2n-4-n+1} - l_{2n-2-n+1}|, |l_{2n-2-n+1} - l_{2n-n+1}|\} \\ &= \{l_2, l_4\} \bigcup \{l_5, l_7\} \bigcup \{l_{n-2}, l_n\}. \end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$

loop for integers $s = 1, 2, \cdots, \dfrac{n-1}{3} - 1$. Let

$$\begin{aligned}
E_3 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{f_1(u_i u_{i+1}) : i = 3s+1\} \\
&= \bigcup_{s=1}^{\frac{n-4}{3}} \{|f(u_i) - f(u_{i+1})| : i = 3s+1\} \\
&= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \cdots, |f(u_{n-3} - f(u_{n-2})|\} \\
&= \{|l_{8-3} - l_{10-6}|, |l_{14-6} - l_{16-9}|, \cdots, |l_{2n-6-n+4} - l_{2n-4-n+1}|\} \\
&= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{n-2} - l_{n-3}|\} = \{l_3, l_6, \cdots, l_{n-4}\}
\end{aligned}$$

Now, $E = \bigcup_{i=1}^{3} E_i = \{l_1, l_2, l_3, l_4, \cdots, l_{n-2}, l_n, l_{n+1}\}$.

Case 2 $n \equiv 2(mod\ 3)$.

Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $a \epsilon N$ by $f(u_1) = l_0$, $f(u_n) = l_{n+2}$. For $s = 1, 2, \cdots, \dfrac{n-2}{3} - 1$, $f(u_i) = l_{2i-3s}$, $3s-1 \leq i \leq 3s+1$ and for $s = \dfrac{n-2}{3}$, $f(u_i) = l_{2i-3s}$, $3s-1 \leq i \leq 3s$. We claim that the edge labels are distinct. Let

$$\begin{aligned}
E_1 &= \{f_1(u_1u_2), f_1(u_{n-1}u_n), f_1(u_nu_1)\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_{n-1}) - f(u_n)|, |f(u_n) - f(u_1)|\} \\
&= \{|l_0 - l_1|, |l_{2n-2-n+2} - l_{n+2}|, |l_{n+2} - l_0|\} = \{l_1, l_{n+1}, l_{n+2}\},
\end{aligned}$$

$$\begin{aligned}
E_2 &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_i u_{i+1}) : 3s-1 \leq i \leq 3s\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\} \\
&= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \bigcup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(u_{n-3}) - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})|\} \\
&= \{|l_{4-3} - l_{6-3}|, |l_{6-3} - l_{8-3}|\} \bigcup \{|l_{10-6} - l_{12-6}|, |l_{12-6} - l_{14-6}|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{2n-6-n+2} - l_{2n-4-n+2}|\} \\
&= \{|l_1 - l_3|, |l_3 - l_5|\} \bigcup \{|l_4 - l_6|, |l_6 - l_8|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{n-4} - l_{n-2}|, |l_{n-2} - l_n|\} \\
&= \{l_2, l_4, l_5, l_7, ..., l_{n-3}, l_{n-1}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of $(s-1)^{th}$ loop and the starting vertex of

s^{th} loop for integers $s = 1, 2, \cdots, \dfrac{n-5}{3}$. Let

$$\begin{aligned} E_3 &= \bigcup_{s=1}^{\frac{n-5}{3}} \{f_1(u_i u_{i+1}) : i = 3s+1\} \\ &= \bigcup_{s=1}^{\frac{n-5}{3}} \{|f(u_i) - f(u_{i+1})| : i = 3s+1\} \\ &= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, ..., |f(u_{n-4}) - f(u_{n-3})|\} \\ &= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{2n-8-n+5} - l_{2n-6-n+2}|\} = \{l_3, l_6, \cdots, l_{n-2}\} \end{aligned}$$

Now, $E = \bigcup_{i=1}^{3} E_i = \{l_1, l_2, l_3, l_4, \cdots, l_{n-3}, l_{n-2}, l_{n-1}, l_{n+1}, l_{n+2}\}$ So, all these edge labels of G are distinct. In both the cases, f is a nearly Lucas graceful graph. Thus $G = C_n$ is a nearly Lucas graceful graph when $n \equiv 1, 2(mod\ 3)$. □

Example 3.5 A nearly Lucas graceful labeling on C_{13} in Case 1 is shown in Fig.3.2.

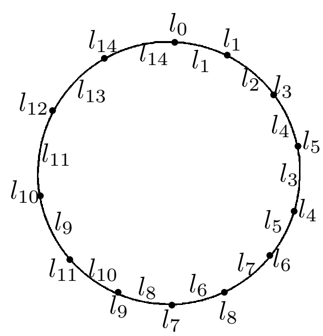

Fig.3.2 C_{13}

Example 3.6 A nearly Lucas graceful labeling on C_{14} in Case 2 is shown in Fig.3.3.

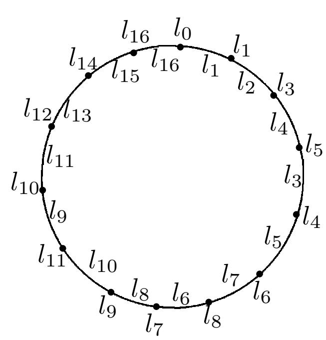

Fig.3.3 C_{14}

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Math.Combin.Book Ser. Vol.3 (2011), 107-113

New Mean Graphs

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Abstract: A vertex labeling of G is an assignment $f : V(G) \to \{1, 2, 3, \ldots, p + q\}$ be an injection. For a vertex labeling f, the induced Smarandachely edge m-labeling f_S^* for an edge $e = uv$, an integer $m \geq 2$ is defined by $f_S^*(e) = \left\lceil \dfrac{f(u) + f(v)}{m} \right\rceil$. Then f is called a Smarandachely super m-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. Particularly, in the case of $m = 2$, we know that

$$f^*(e) = \begin{cases} \dfrac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \dfrac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Such a labeling is usually called a mean labeling. A graph that admits a Smarandachely super mean m-labeling is called a Smarandachely super m-mean graph, particularly, a mean graph if $m = 2$. In this paper, some new families of mean graphs are investigated. We prove that the graph obtained by two new operations called mutual duplication of a pair of vertices each from each copy of cycle C_n as well as mutual duplication of a pair of edges each from each copy of cycle C_n admits mean labeling. More over that mean labeling for shadow graphs of star $K_{1,n}$ and bistar $B_{n,n}$ are derived.

Key Words: Smarandachely super m-mean labeling, mean labeling, Smarandachely super m-mean graph, mean graphs; mutual duplication.

AMS(2010): 05C78

§1. Introduction

We begin with simple,finite,connected and undirected graph $G = (V(G), E(G))$ with p vertices and q edges. For all other standard terminology and notations we follow Harary [3]. We will provide brief summary of definitions and other information which serve as prerequisites for the present investigations.

[1]Received February 21, 2011. Accepted September 8, 2011.

Definition 1.1 *Consider two copies of cycle C_n. Then the mutual duplication of a pair of vertices v_k and v'_k respectively from each copy of cycle C_n produces a new graph G such that $N(v_k) = N(v'_k)$.*

Definition 1.2 *Consider two copies of cycle C_n and let $e_k = v_k v_{k+1}$ be an edge in the first copy of C_n with $e_{k-1} = v_{k-1}v_k$ and $e_{k+1} = v_{k+1}v_{k+2}$ be its incident edges. Similarly let $e'_m = u_m u_{m+1}$ be an edge in the second copy of C_n with $e'_{m-1} = u_{m-1}u_m$ and $e'_{m+1} = u_{m+1}u_{m+2}$ be its incident edges. The mutual duplication of a pair of edges e_k, e'_m respectively from two copies of cycle C_n produces a new graph G in such a way that $N(v_k) - v_{k+1} = N(u_m) - u_{m+1}$ $=\{v_{k-1}, u_{m-1}\}$ and $N(v_{k+1}) - v_k = N(u_{m+1}) - u_m$ $=\{v_{k+2}, u_{m+2}\}$.*

Definition 1.3 *The shadow graph $D_2(G)$ of a connected graph G is obtained by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.*

Definition 1.4 *Bistar is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$ by an edge.*

Definition 1.5 *If the vertices are assigned values subject to certain conditions then it is known as graph labeling.*

Graph labeling is one of the fascinating areas of research with wide ranging applications. Enough literature is available in printed and electronic form on different types of graph labeling and more than 1200 research papers have been published so far in past four decades. Labeled graph plays vital role to determine optimal circuit layouts for computers and for the representation of compressed data structure. For detailed survey on graph labeling we refer to *A Dynamic Survey of Graph Labeling* by Gallian [2]. A systematic study on various applications of graph labeling is carried out in Bloom and Golomb [1].

Definition 1.6 *A vertex labeling of G is an assignment $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For a vertex labeling f, the induced Smarandachely edge m-labeling f^*_S for an edge $e = uv$, an integer $m \geq 2$ is defined by $f^*_S(e) = \left\lceil \dfrac{f(u)+f(v)}{m} \right\rceil$. Then f is called a Smarandachely super m-mean labeling if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. Particularly, in the case of $m = 2$, we know that*

$$f^*(e) = \begin{cases} \dfrac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even;} \\ \dfrac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

Such a labeling is usually called a mean labeling. A graph that admits a Smarandachely super mean m-labeling is called a Smarandachely super m-mean graph, particularly, a mean graph if $m = 2$.

The mean labeling was introduced by Somasundaram and Ponraj [4] and they proved the graphs P_n, C_n, $P_n \times P_m$, $P_m \times C_n$ etc. admit mean labeling. The same authors in [5] have discussed the mean labeling of subdivision of $K_{1,n}$ for $n < 4$ while in [6] they proved that the

wheel W_n does not admit mean labeling for $n > 3$. Mean labeling in the context of some graph operations is discussed by Vaidya and Lekha[7] while in [8] the same authors have investigated some new families of mean graphs. In the present work four new results corresponding to mean labeling are investigated.

§2. Main Results

Theorem 2.1 *The graph obtained by the mutual duplication of a pair of vertices in cycle C_n admits mean labeling.*

Proof Let v_1,v_2,..., v_n be the vertices of the first copy of cycle C_n and let u_1,u_2,..., u_n be the vertices of the second copy of cycle C_n. Let G be the graph obtained by the mutual duplication of a pair of vertices each respectively from each copy of cycle C_n. To define $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ two cases are to be considered.

Case 1. n is odd.

Without loss of generality assume that the vertex $v_{\frac{n+3}{2}}$ from the first copy of cycle C_n and the vertex u_1 from the second copy of cycle C_n are mutually duplicated.

$$f(v_i) = 2i - 2 \text{ for } 1 \leq i \leq \frac{n+1}{2},;$$
$$f(v_i) = 2(n - i) + 3 \text{ for } \frac{n+3}{2} \leq i \leq n;$$
$$f(u_1) = n + 4;$$
$$f(u_i) = n + 2i + 3 \text{ for } 2 \leq i \leq \frac{n+1}{2};$$
$$f(u_i) = 3n - 2i + 6 \text{ for } \frac{n+3}{2} \leq i \leq n.$$

Case 2: n is even.

Without loss of generality assume that the vertex $v_{\frac{n+2}{2}}$ from the first copy of cycle C_n and the vertex u_1 from the second copy of cycle C_n are mutually duplicated.

$$f(v_i) = 2i - 2 \text{ for } 1 \leq i \leq \frac{n+2}{2};$$
$$f(v_i) = 2(n - i) + 3 \text{ for } \frac{n+4}{2} \leq i \leq n;$$
$$f(u_1) = n + 4;$$
$$f(u_i) = n + 2i + 3 \text{ for } 2 \leq i \leq \frac{n}{2};$$
$$f(u_i) = 3n - 2i + 6 \text{ for } \frac{n+2}{2} \leq i \leq n.$$

In view of the above defined labeling pattern f is a mean labeling for the graph obtained by the mutual duplication of a pair of vertices in cycle C_n. □

Illustration 2.2 The following Fig.1 shows the pattern of mean labeling of the graph obtained by the mutual duplication of a pair of vertices of cycle C_{10}.

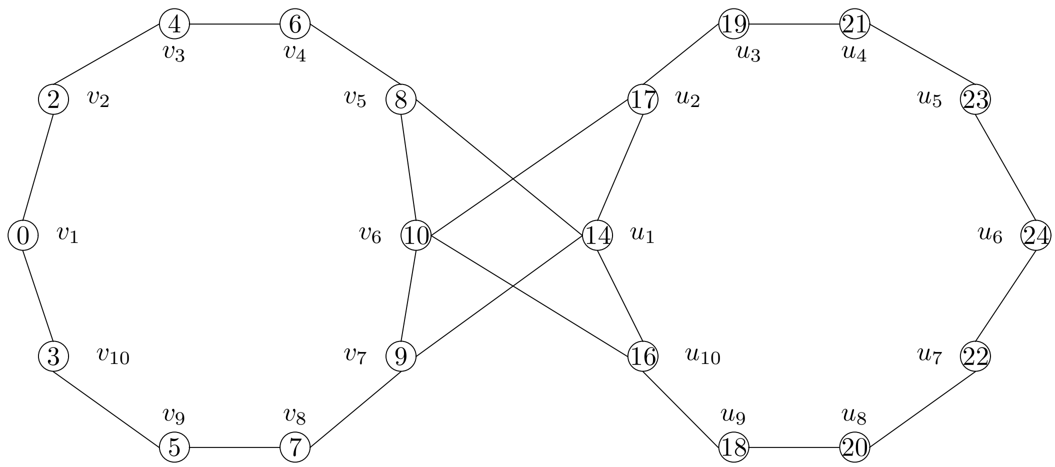

Fig.1

Theorem 2.3 *The graph obtained by the mutual duplication of a pair of edges in cycle C_n admits mean labeling.*

Proof Let v_1,v_2,..., v_n be the vertices of the first copy of cycle C_n and let u_1,u_2,..., u_n be the vertices of the second copy of cycle C_n. Let G be the graph obtained by the mutual duplication of a pair of edges each respectively from each copy of cycle C_n. To define $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ two cases are to be considered.

Case 1. n is odd.

Without loss of generality assume that the edge $e= v_{\frac{n+1}{2}}v_{\frac{n+3}{2}}$ from the first copy of cycle C_n and the edge $e'= u_1u_2$ from the second copy of cycle C_n are mutually duplicated.

$f(v_1) = 0;$

$f(v_i) = 2i - 1$ for $2 \leq i \leq \dfrac{n+1}{2};$

$f(v_i) = 2(n - i) + 2$ for $\dfrac{n+3}{2} \leq i \leq n;$

$f(u_i) = n + 2i + 2$ for $1 \leq i \leq \dfrac{n+1}{2};$

$f(u_i) = 3n - 2i + 7$ for $\dfrac{n+3}{2} \leq i \leq n.$

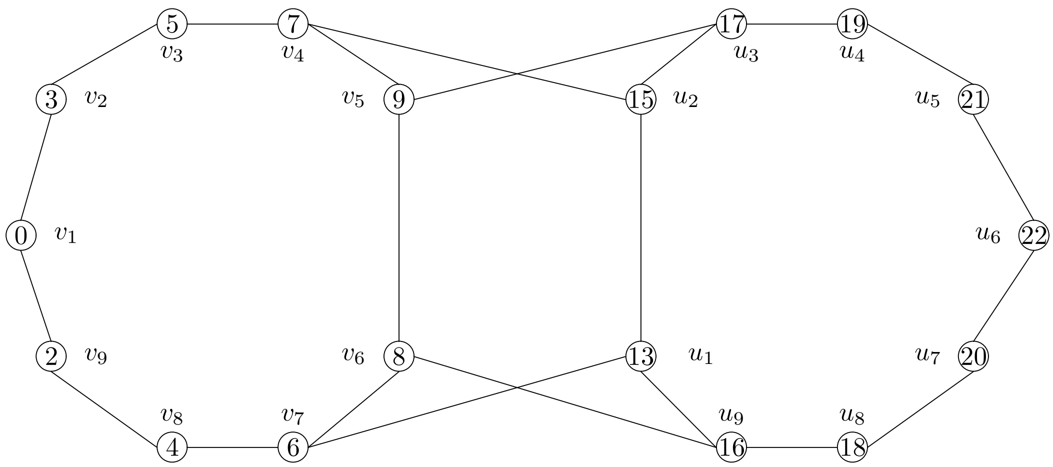

Fig.2

Case 2. n is even, $n \neq 4$.

Without loss of generality assume that the edge $e = v_{\frac{n}{2}+1}v_{\frac{n}{2}+2}$ from the first copy of cycle C_n and the edge $e' = u_1u_2$ from the second copy of cycle C_n are mutually duplicated.

$f(v_i) = 2i - 2$ for $1 \leq i \leq \dfrac{n}{2} + 1$;
$f(v_i) = 2(n - i) + 3$ for $\dfrac{n}{2} + 2 \leq i \leq n$;
$f(u_i) = n + 2i + 2$ for $1 \leq i \leq \dfrac{n}{2} + 1$;
$f(u_i) = 3n - 2i + 7$ for $\dfrac{n}{2} + 2 \leq i \leq n$.

Then above defined function f provides mean labeling for the graph obtained by the mutual duplication of a pair of edges in C_n. □

Illustration 2.4 The following Fig.2 shows mean labeling for the graph obtained by the mutual duplication of a pair of edges in cycle C_9.

Theorem 2.5 *$D_2(K_{1,n})$ is a mean graph.*

Proof Consider two copies of $K_{1,n}$. Let $v, v_1, v_2, \ldots, v_n$ be the vertices of the first copy of $K_{1,n}$ and $v', v'_1, v'_2, \ldots, v'_n$ be the vertices of the second copy of $K_{1,n}$ where v and v' are the respective apex vertices. Let G be $D_2(K_{1,n})$. Define $f : V(G) \to \{0, 1, 2, \ldots, q\}$ as follows.

$f(v) = 0$;
$f(v_i) = 2i$ for $1 \leq i \leq n$;
$f(v') = 4n$;
$f(v'_1) = 4n - 1$;
$f(v'_i) = 4n - 2i + 2$ for $2 \leq i \leq n$.

The above defined function provides the mean labeling of the graph $D_2(K_{1,n})$. □

Illustration 2.6 The labeling pattern for $D_2(K_{1,4})$ is given in Fig.3.

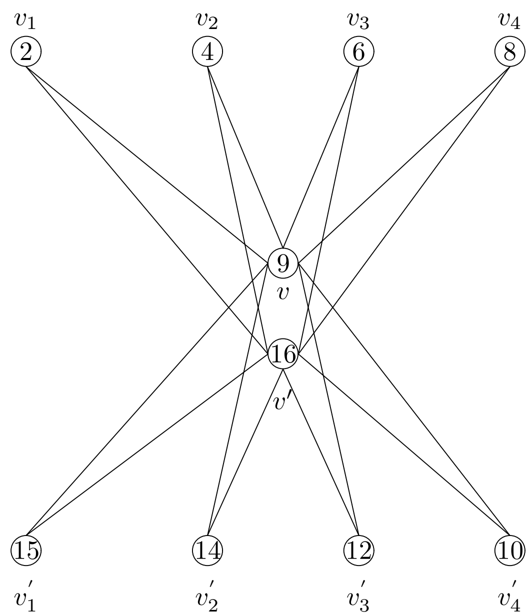

Fig.3

Theorem 2.7 *$D_2(B_{n,n})$ is a mean graph.*

Proof Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ and $\{u', v', u_i', v_i', 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Let G be $D_2(B_{n,n})$. Define $f: V(G) \to \{0, 1, 2, \ldots, q\}$ as follows.

$f(u) = 0$;

$f(u_i) = 2i$ for $1 \leq i \leq n$;

$f(v) = 8n + 1$;

$f(v_i) = 4i + 1$ for $1 \leq i \leq n - 1$;

$f(v_n) = 4n + 5$;

$f(u') = 4n$;

$f(u_i') = 2(n + i)$ for $1 \leq i \leq n - 1$;

$f(u_n') = 4n - 1$;

$f(v') = 8n + 3$;

$f(v_i') = 8(n + 1) - 4i$ for $1 \leq i \leq n$.

In view of the above defined labeling pattern G admits mean labeling. □

Illustration 2.8 The labeling pattern for $D_2(B_{3,3})$ is given in Fig.4.

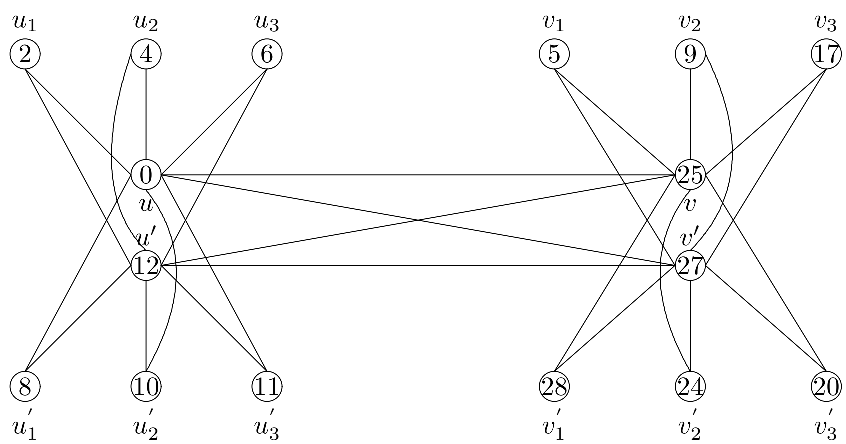

Fig.4

§3. Concluding Remarks

As all the graphs are not mean graphs it is very interesting to investigate graphs or graph families which admit mean labeling. Here we contribute two new graph operations and four new families of mean graphs. Somasundaram and Ponraj have proved that star $K_{1,n}$ is mean graph for $n \leq 2$ and bistar $B_{m,n}$ $(m > n)$ is mean graph if and only if $m < n + 2$ while in this paper we have investigated that the shadow graphs of star $K_{1,n}$ and bistar $B_{n,n}$ also admit mean labeling.

To investigate similar results for other graph families and in the context of different labeling techniques is an open area of research.

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Vertex-Mean Graphs

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Abstract: Let $k \geq 0$ be an integer. A *Smarandachely vertex-mean k-labeling* of a (p,q) graph $G=(V,E)$ is such an injection $f: E \longrightarrow \{0,1,2,...,q_*+k\}$, $q_* = \max(p,q)$ such that the function $f^V: V \longrightarrow \mathbb{N}$ defined by the rule $f^V(v) = Round\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right) - k$ satisfies the property that $f^V(V) = \left\{f^V(u) : u \in V\right\} = \{1,2,...,p\}$, where E_v denotes the set of edges in G that are incident at v, $\mathbb{N}$ denotes the set of all natural numbers and *Round* is the *nearest integer function*. A graph that has a Smarandachely vertex-mean k-labeling is called *Smarandachely k vertex-mean graph* or *Smarandachely k V-mean graph*. Particularly, if $k=0$, such a Smarandachely vertex-mean 0-labeling and Smarandachely 0 vertex-mean graph or Smarandachely 0 V-mean graph is called a *vertex-mean labeling* and a *vertex-mean graph* or *V-mean graph*, respectively. In this paper, we obtain necessary conditions for a graph to be V-mean and study V-mean behaviour of certain classes of graphs.

Key Words: Smarandachely vertex-mean k-labeling, vertex-mean labeling, edge labeling, Smarandachely k vertex-mean graph, vertex-mean graph.

AMS(2010): 05C78

§1. Introduction

A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces a label for each edge xy depending on the vertex labels. An *edge labeling* of a graph G is an assignment f of labels to the edges of G that induces a label for each vertex v depending on the labels of the edges incident on it. Vertex labelings such as *graceful labeling, harmonious labeling* and *mean labeling* and edge labelings such as *edge-magic labeling, (a,d)-anti magic labeling and vertex-graceful labeling* are some of the interesting labelings found in the dynamic survey of graph labeling by Gallian [3]. In fact B. D. Acharya [2] has introduced *vertex-graceful graphs*, as an edge-analogue of *graceful graphs*. Observe that, in a variety of practical problems, the arithmetic mean, X, of a finite set of real numbers $\{x_1, x_2, ..., x_n\}$ serves as a

[1]Received February 12, 2011. Accepted September 10, 2011.

better estimate for it, in the sense that $\sum(x_i - X)$ is zero and $\sum(x_i - X)^2$ is the minimum. If it is required to use a single integer in the place of X then $Round(X)$ does this best, in the sense that $\sum(x_i - Round(X))$ and $\sum(x_i - Round(X))^2$ are minimum, where $Round(Y)$, *nearest integer function* of a real number, gives the integer closest to Y; to avoid ambiguity, it is defined to be the nearest even integer in the case of half integers. This motivates us to define the edge-analogue of the *mean labeling* introduced by R. Ponraj [1]. A *mean labeling* f is an injection from V to the set $\{0, 1, 2, ..., q\}$ such that the set of edge labels defined by the rule $Round(\frac{f(u)+f(v)}{2})$ for each edge uv is $\{1, 2, ..., q\}$. For all terminology and notations in graph theory, we refer the reader to the text book by D. B. West [4]. All graphs considered in the paper are finite and simple.

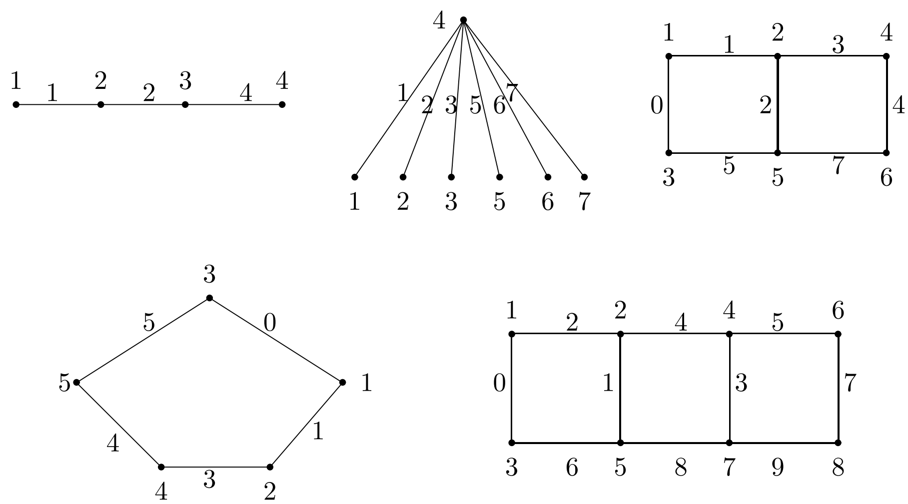

Fig.1 **Some V -mean graphs**

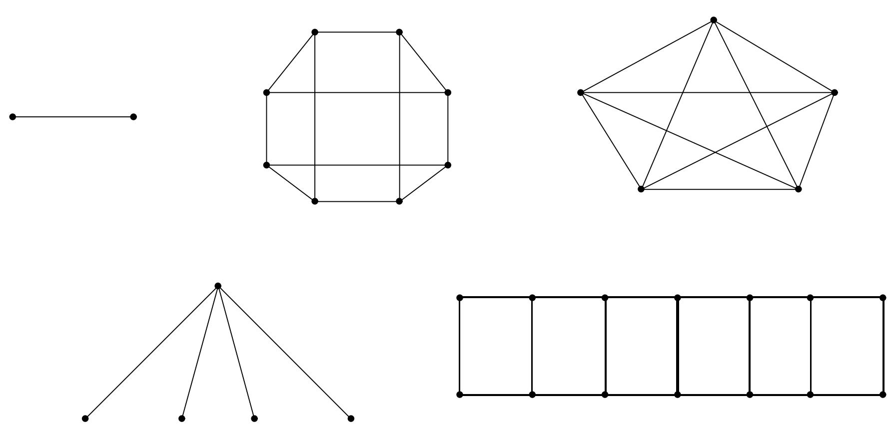

Fig.2

Definition 1.1 *Let $k \geq 0$ be an integer. A Smarandachely vertex-mean k-labeling of a (p,q) graph $G=(V,E)$ is such an injection $f: E \longrightarrow \{0,1,2,...,q_*+k\}$, $q_* = \max(p,q)$ such that the function $f^V: V \longrightarrow \mathbb{N}$ defined by the rule $f^V(v) = Round\left(\frac{\sum_{e\in E_v} f(e)}{d(v)}\right) - k$ satisfies the property that $f^V(V) = \left\{f^V(u): u \in V\right\} = \{1,2,...,p\}$, where E_v denotes the set of edges in G that are incident at v, $\mathbb{N}$ denotes the set of all natural numbers and Round is the nearest integer function. A graph that has a Smarandachely vertex-mean k-labeling is called Smarandachely k vertex-mean graph or Smarandachely k V-mean graph. Particularly, if $k=0$, such a Smarandachely vertex-mean 0-labeling and Smarandachely 0 vertex-mean graph or Smarandachely 0 V-mean graph is called a vertex-mean labeling and a vertex-mean graph or V-mean graph, respectively.*

Henceforth we call vertex-mean as V-mean. To initiate the investigation, we obtain necessary conditions for a graph to be a V-mean graph and we present some results on this new notion in this paper. In Fig.1 we give some V-mean graphs and in Fig.2, we give some non V-mean graphs.

§2. Necessary Conditions

Following observations are obvious from Definition 1.1.

Observation 2.1 If G is a V-mean graph then no V-mean labeling assigns 0 to a pendant edge.

Observation 2.2 The graph K_2 and disjoint union of K_2 are not V-mean graphs, as any number assigned to an edge uv leads to assignment of same number to each of u and v. Thus every component of a V-mean graph has at least two edges.

Observation 2.3 The minimum degree of any V-mean graph is less than or equal to three ie, $\delta \leqslant 3$ as $Round(0+1+2+3)$ is 2. Thus graphs that contain a r-regular graph, where $r \geq 4$ as spanning sub graph are not V-mean graphs and any 3-edge-connected V-mean graph has a vertex of degree three.

Observation 2.4 If f is a V-mean labeling of a graph G then either (1) or (2) of the following is satisfied according as the induced vertex label $f^V(v)$ is obtained by rounding up or rounding down.

$$f^V(v)d(v) \leq \sum_{e\in E_v} f(e) + \frac{1}{2}d(v), \tag{1}$$

$$f^V(v)d(v) \geq \sum_{e\in E_v} f(e) - \frac{1}{2}d(v). \tag{2}$$

Theorem 2.5 *If G is a V-mean graph then the vertices of G can be arranged as $v_1, v_2, ..., v_p$ such that $q^2 - 2q \leq \sum_1^p kd(v_k) \leq 2qq_* - q^2 + 2q$.*

Proof Let f be a V-mean labeling of a graph G. Let us denote the vertex that has the induced vertex label k, $1 \leq k \leq p$ as v_k. Observe that, $\sum_{v\in V} f^V(v)d(v)$ attains it maximum/minimum when each induced vertex label is obtained by rounding up/down and the first

q largest/smallest values of the set $\{0, 1, 2, \cdots, q_*\}$ are assigned as edge labels by f. This with Observation 2.4 completes the proof. □

Corollary 2.6 *Any 3-regular graph of order 2m, $m \geq 4$ is not a V-mean graph.*

Corollary 2.7 *The ladder $L_n = P_n \times P_2$, $n \geq 7$ is not a V-mean graph.*

A V-mean labeling of ladders L_3 and L_4 are shown in Figure 1.

§3. Classes of V-Mean Graphs

Theorem 3.1 *If $n \geq 3$ then the path P_n is V-mean graph.*

Proof Let $\{e_1, e_2, ..., e_{n-1}\}$ be the edge set of P_n such that $e_i = v_i v_{i+1}$. We define $f : E \longrightarrow \{0, 1, 2, ..., q_* = p\}$ as follows:

$$f(e_i) = \begin{cases} i, & if\ 1 \leq i \leq p-2, \\ i+1, & if\ i = p-1. \end{cases}$$

It can be easily verified that f is a V-mean labeling. □

A V-mean labeling of P_{10} is shown in Fig.3.

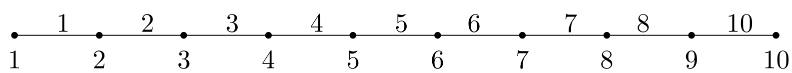

Fig.3

Theorem 3.2 *If $n \geq 3$ then the cycle C_n is V-mean graph.*

Proof Let $\{e_1, e_2, ..., e_n\}$ be the edge set of C_n such that $e_i = v_i v_{i+1}$, $1 \leq i \leq n-1$, $e_n = v_n v_1$. Let $\zeta = \lceil \frac{n}{2} \rceil - 1$. The edges of C_n are labeled as follows: The numbers $0, 1, 2, \cdots, n$ except ζ are arranged in an increasing sequence $\alpha_1, \alpha_2, \cdots, \alpha_n$ and α_k is assigned to e_k. Clearly the edges of C_n receive distinct labels and the vertex labels induced are $1, 2, \cdots, n$. Thus C_n is V-mean graph. □

The *corona* $G_1 \odot G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2. The graph $C_n \odot K_1$ is called a *crown.*

Theorem 3.3 *The corona $P_n \odot K_m^C$, where $n \geq 2$ and $m \geq 1$ is V-mean graph.*

Proof Let the vertex set and the edge set of $G = P_n \odot K_m^C$ be as follows:

$V(G) = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n \ and \ 1 \leq j \leq m\}$,

$E(G) = A \bigcup B$,

where $A = \{e_i = u_iu_{i+1} : 1 \leq i \leq n-1\}$ and $B = \{e_{ij} = u_iu_{ij} : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$. We observe that G has order $(m+1)n$ and size $(m+1)n-1$. The edges of G are labeled in three steps as follows :

Step 1. The edges e_1 and $e_{1j}, 1 \leq j \leq m$ are assigned distinct integers from 1 to $(m+1)$ in such a way that e_1 receives the number $Round(\frac{\sum_{j=1}^{m+1} j}{m+1})$.

Step 2. For each i, $2 \leq i \leq n-1$, the edges e_i and $e_{ij}, 1 \leq j \leq m$ are assigned distinct integers from $(m+1)(i-1)+1$ to $(m+1)i$ in such a way that e_i receives the number

$$Round(\frac{f(e_{i-1}) + \sum_{j=1}^{m+1}(m+1)(i-1)+j}{m+2}).$$

Step 3. The edges $e_{nj}, 1 \leq j \leq m$ are assigned distinct integers from $(m+1)(n-1)+1$ to $(m+1)n$ in such a way that non of these edges receive the number

$$Round(\frac{f(e_{n-1}) + \sum_{j=1}^{m+1}(m+1)(n-1)+j}{m+2}).$$

Then the edges of G receive distinct labels and the vertex labels induced are $1, 2, ..., (m+1)n$. Thus G is V-mean graph.

Fig.4 displays a V-mean labeling of $P_5 \odot K_4^C$.

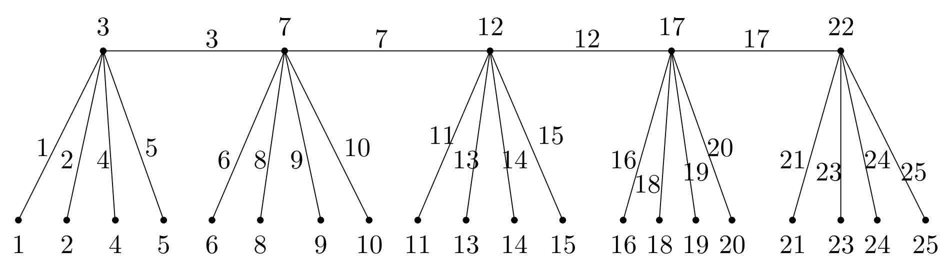

Fig.4 **A V -mean labeling of $P_5 \odot K_4^C$**

Theorem 3.4 *The star graph $K_{1,n}$ is V-mean graph if and only if $n \cong 0(mod 2)$.*

Proof Necessity: Suppose $G = K_{1,n}$, $n = 2m+1$ for some $m \geq 1$ is V-mean and let f be a V-mean labeling of G. As no V-mean labeling assigns zero to a pendant edge, f assigns $2m+1$ distinct numbers from the set $\{1, 2, ..., 2m+2\}$ to the edges of G. Observe that, whatever be the labels assigned to the edges of G, label induced on the central vertex of G will be either $m+1$ or $m+2$. In both cases two vertex labels induced on G will be identical. This contradiction proves *necessity.*

Sufficiency: Let $G = K_{1,n}$, $n = 2m$ for some $m \geq 1$. Then assignment of $2m$ distinct numbers except $m+1$ from the set $\{1, 2, ..., 2m+1\}$ gives the desired V-mean labeling of G.□

Theorem 3.5 *The crown $C_n \odot K_1$ is V-mean graph.*

Proof Let the vertex set and the edge set of $G = C_n \odot K_1$ be as follows: V(G) $= \{u_i, v_i : 1 \le i \le n\}$, $E(G) = A \bigcup B$ where $A = \{e_i = u_iu_{i+1} : 1 \le i \le n-1\} \bigcup \{e_n = u_nu_1\}$ and $B = \left\{e_i^{'} = u_iv_i : \ 1 \le i \le n\right\}$. Observe that G has order and size both equal to $2n$. For $3 \le n \le 5$, V-mean labeling of G are shown in Fig.5. For $n \ge 6$, define $f : E(G) \longrightarrow \{0, 1, 2, ..., 2n\}$ as follows:

Case 1 $n \equiv 0(mod\ 3)$.

$$f(e_i) = \begin{cases} 2i-2 & if\ 1 \le i \le \dfrac{n}{3} - 1, \\ 2i & if\ i = \dfrac{n}{3}, \\ 2i-1 & if\ \dfrac{n}{3} + 1 \le i \le n, \end{cases}$$

$$f(e_i^{'}) = \begin{cases} 2i-1 & if\ 1 \le i \le \dfrac{n}{3}, \\ 2i & if\ \dfrac{n}{3} + 1 \le i \le n. \end{cases}$$

Case 2 $n \not\equiv 0(mod\ 3)$.

$$f(e_i) = \begin{cases} 2i-2 & if\ 1 \le i \le \left\lfloor \dfrac{n}{3} \right\rfloor, \\ 2i-1 & if\ \left\lfloor \dfrac{n}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(e_i^{'}) = \begin{cases} 2i-1 & if\ 1 \le i \le \left\lfloor \dfrac{n}{3} \right\rfloor, \\ 2i & if\ \left\lfloor \dfrac{n}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

It can be easily verified that f is a V-mean labeling of G. $\square$

A V-mean labeling of some crowns are shown in Fig.5.

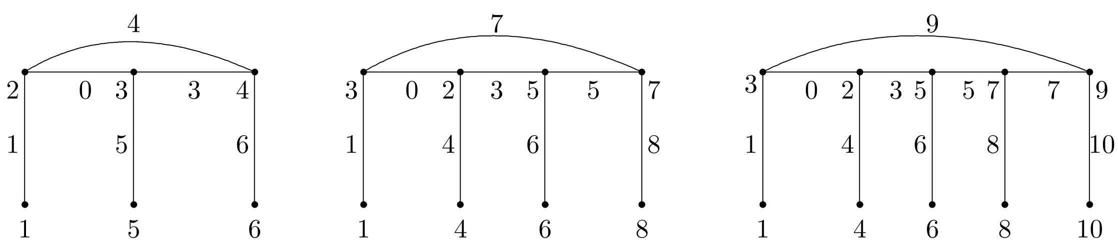

Fig.5 V **-mean labeling of crowns for** $n = 3, 4, 5$

Problem 3.6 *Determine new classes of trees and unicyclic graphs which are V-mean graphs.*

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The history of mankind is the history of ideas.

By Ludwig Von Mises, an Austrian-American economist and philosopher.

Author Information

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[12]W.S. Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).

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Mathematical Combinatorics (*International Book Series*) September 2011

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ISBN 9781599731643
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